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THE VISIBLE RADIATION FROM CARBON. II.

By Edward L. Nichols.

Spectro-photometric Observations.

IT was my expectation, in planning this research that whatever might prove true as to the character of the radiation from gray carbon, the distribution of energy in the spectrum black carbon would change in such a manner with increasing incandescence as to become nearly or quite identical with that of the various luminous gas flames at temperatures corresponding to the temperature of the glowing carbon in those flames. I had also hoped, among other things, to be able to bring about a degree of incandescence approaching that of the acetylene flame itself, before the usefulness of the thermo-element as a means of measuring the temperature failed because of the melting of the platinum wire, and in this way to obtain a check upon my previous measurements of that flame, and at the same time to be able to determine the temperature of any given luminous flame in which the incandescent material consists of carbon particles by ascertaining the temperature of the carbon rod for which its surface had a spectrum corresponding in distribution of energy to that of the flame.

It will be seen from inspection of the curves to be discussed in a subsequent paragraph that this expectation was far from being realized, and that the distribution of energy in the spectrum of the carbon rod instead of approaching that of the acetylene flame as the temperature of the rod increased took on an entirely unexpected

character. Even at low temperatures, that is to say up to about 1100° the change in the spectrum was not of the comparatively simple character which had been anticipated, and shortly after passing the temperature of 1100° a very surprising complication in the results arose. The energy in the yellow of the spectrum, which from the beginning had been increasing at a relatively more rapid rate than either in the red or at the blue end, became so great as to give to the distribution curve a form entirely contrary to expectation.

I was very slow to believe in the integrity of these results and nearly a year was spent in repetitions of the measurements before I could convince myself that the phenomenon was a genuine one. Measurements taken upon a great number of different rods and at different times showed the same result, however, and I was finally forced to the conclusion that the radiation from the carbon rods showed a much more complicated law of distribution than had been anticipated, and that a sort of selective radiation occurred such as to render the establishing of any simple relationship between the curve of distribution and temperature out of the question.

The hope of being able to make direct temperature measurements up to the melting point of platinum was also disappointed. While the carbon rods at comparatively low temperatures showed a fair degree of stability under the action of the current, they appeared to undergo a decided change of behavior at about 1500° and before that temperature a rather rapid disintegration, showing itself by a change of resistance, manifested itself. This effect appeared to be similar to that which shortens the life of the filaments of incandescent lamps. It appeared also that at these high temperatures, the carbon tends to combine with the metals of the thermo-element affecting the electromotive force very much as the vapors in a furnace have been found to do. The thermo-elements inserted in the rod begin, in consequence of this action, to fail of their purpose. It was found that after exposure to temperatures much above 1400°, the electromotive force corresponding to even lower temperatures was considerably below the normal. I was consequently compelled to abandon the attempt to measure directly temperatures above this point, although it was possible to bring the rods to a higher degree

of incandescence for a length of time sufficient to perform the spectro-photometric observations. In order to obtain at least an approximate estimate of these high temperatures, I made use of the fall of potential between the terminals of the rod, and also of the current of the heating circuit, and by extending these curves which, throughout the range of measured temperatures were found to be nearly straight, to the higher temperatures which I wished to estimate, to obtain some idea, even if not an exact one, of the latter.

In expressing the results of the photometric measurements already described, I have made use of two forms of curve. One set of curves, in accordance with the nomenclature proposed in my original paper on the visible radiation from platinum, and later adopted by Paschen[1] and other writers, I may designate *isotherms*. These curves give, in terms of the corresponding wave-lengths of the comparison source (in this case the acetylene flame), the relative distribution of energy in the visible radiation from carbon rods. The other curves, which I have termed *isochroms*, indicate the rise in the energy of any particular wave-length of the visible spectrum, with increase of temperature. Each of these curves, taken by itself, is entirely independent of the nature of the light of the comparison source but the absolute relation of such curves to one another can only be obtained when we know the distribution of energy in the spectrum of that source. By means of the isochroms, it is, however, possible even without this knowledge, to compare the rise in intensity of various wave-lengths of the spectrum with increasing temperature.

The set of curves shown in Fig. 7 are plotted directly from observations upon a black (untreated) carbon at temperatures ranging between 795° C. and 1055° C. In this diagram abscissæ are wave-lengths and ordinates are ratios of the brightness of the spectrum of the carbon rod in each region to that of the corresponding region in the spectrum of the acetylene flame. The data are given in Table I.

A noteworthy fact exhibited by means of these curves is the relatively rapid increase of intensity in the middle of the spectrum. In

[1] Paschen, Wiedemann's Annalen, LVIII., p. 455, 1896.

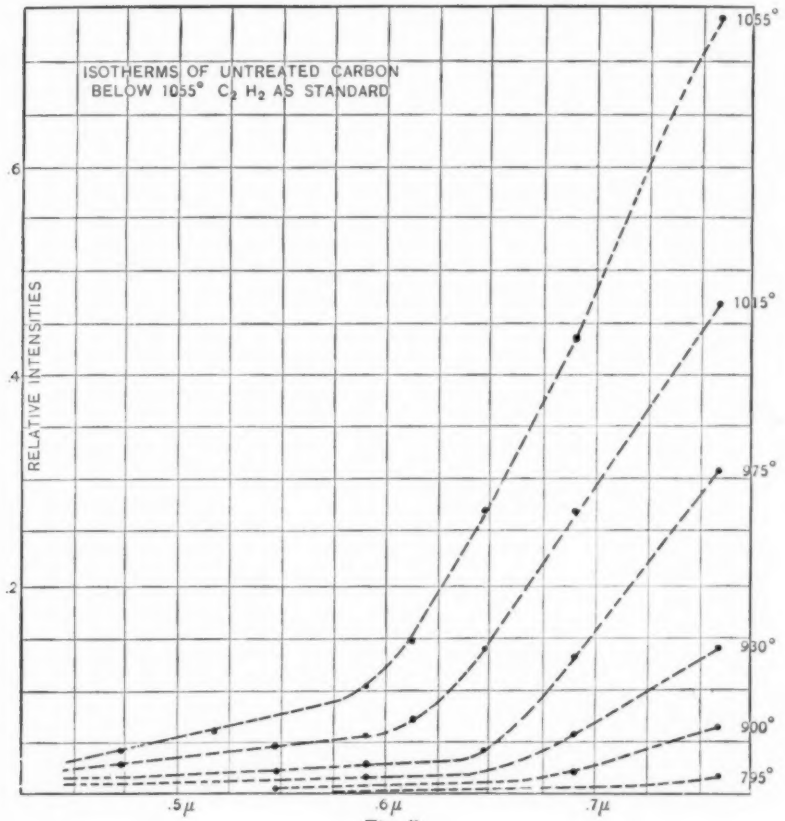

Fig. 7.

TABLE I.

Relative intensities of various regions in the spectrum of an untreated carbon rod; the intensity of the corresponding region in the spectrum of the acetylene flame being in such case taken as unity.

	Value of $\frac{\text{Intensity of Carbon}}{\text{Intensity of } C_2H_2}$ at Temperatures Given Below.					
λ	795°C.	9000C.	930°C.	975°C.	1015°C.	1055°C.
.760μ	.018	.064	.140	.310	.466	.742
.690μ	.007	.022	.057	.131	.270	.436
.647μ	.005	.012	.022	.042	.128	.272
.613μ					.072	.148
.589μ	.003	.008	.017	.028	.067	.103
.548μ			.013	.022	.046	
.518μ			.011	.017		.058
.470μ			.008	.014	.027	.041
.440μ			.006	.012	.022	.030

passing from 930° to 1055° the brightness of wave-length .76 μ increases 5.3 times; that of .70 μ, 7.2 times; that of .60 μ, 13.5 times

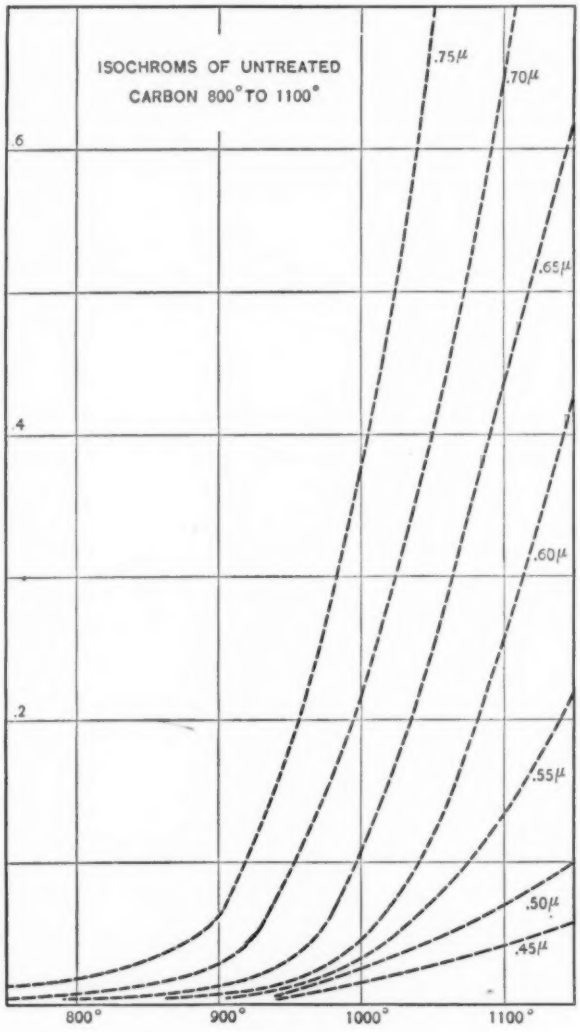

Fig. 8.

and that of .50 μ only 9 times. We have here the beginnings of a process which becomes more marked in its effects as higher temperatures are attained.

From 1100° upwards it was found much more difficult to obtain satisfactory readings. The carbon rods which I had brought from Paris for this investigation would not stand prolonged heating and it was necessary to replace them frequently. In order to bring the

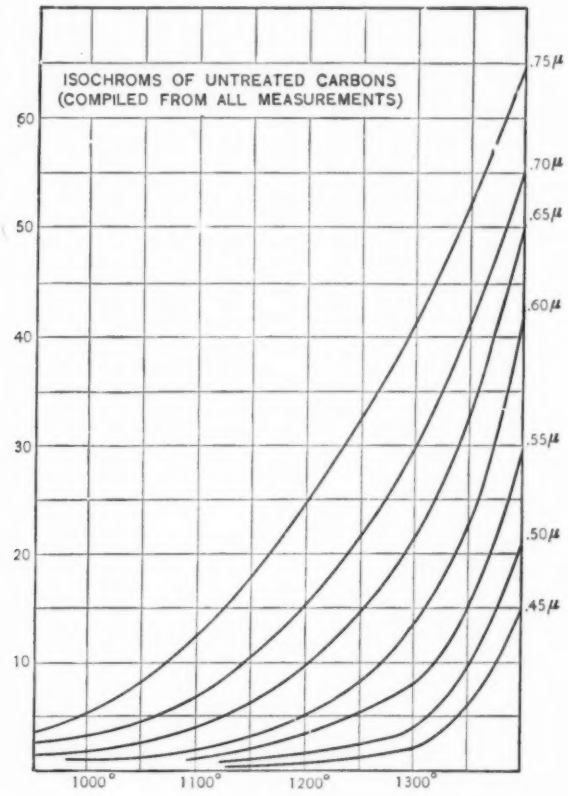

Fig. 9.

observations upon the various rods to a common scale, isochroms from the readings for each rod were plotted. The general character of these curves is shown in Fig. 8 in which the isochroms corresponding to the isotherms of Fig. 7 are given. From the ordinate at 1000° of the isochrom for .6μ, which for convenience was taken as unity for the entire set a reduction factor was obtained by means of which all the curves for all the carbons were brought to the same scale. A new set of isochroms were then plotted for each of the

wave-lengths .75 μ, .70 μ, .65 μ, .60 μ, .55 μ, .50 μ and .45 μ in the drawings of which all the observations upon the rods were used. While this method did not bring the various sets of observations into perfect agreement, the results were sufficiently definite to indicate with a close degree of approximation the trend of these curves for temperatures up to 1400°. The result of this compilation for the wave-lengths just mentioned is shown graphically in Fig. 9. From these curves in turn, isotherms for the temperatures 900°, 1000°, 1100°, 1200°, 1300° and 1400° were plotted. These

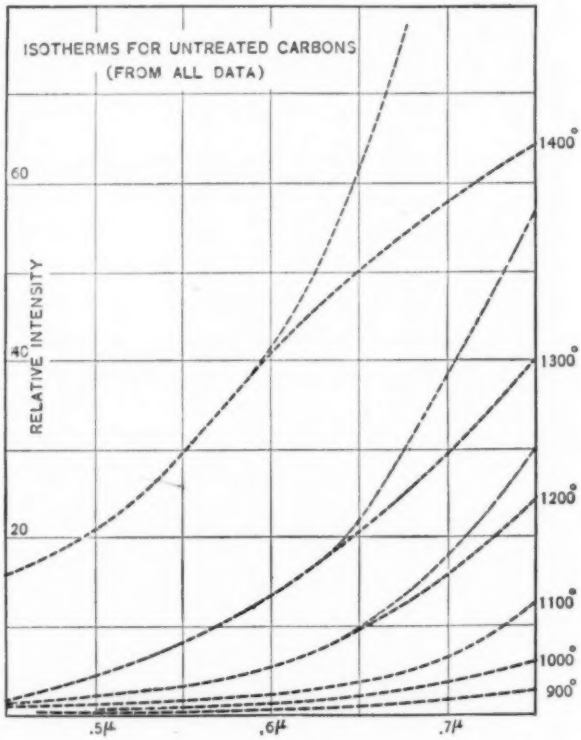

Fig. 10.

curves are given in Fig. 10. The numerical data are contained in Table II.

Had the law of increasing intensities throughout the spectrum with rising temperature been that anticipated at the beginning of

TABLE II.

Relative intensities of various regions in the spectrum of untreated carbon, compiled from all data; the brightness of the spectrum of the acetylene flame being taken as unity throughout and the intensity of the region .6μ in the spectrum of carbon being taken as equal in brightness to that of the corresponding region in the flame spectrum when the temperature of the rod is 1000° *C.*

	Value of $\frac{\text{Intensity of Carbon}}{\text{Intensity of } C_2H_2}$ at Temperatures Given Below.					
λ	1400°C.	1300°C.	1200°C.	1100°C.	1000°C.	900°C.
.750μ	64.4	40.4	24.2	11.9	5.2	2.3
.700μ	58.2	29.7	15.6	6.0	3.2	1.6
.650μ	50.3	21.0	9.8	3.4	1.7	0.8
.600μ	40.7	13.6	5.3	2.1	1.0	0.5
.550μ	29.7	8.2	3.3	1.5	0.5	0.3
.500μ	21.3	4.2	2.2	1.2	0.3	
.450μ	16.3	1.8	1.3	0.7		

this investigation, the trend of the isochroms would necessarily have been such as to bring all the curves together at a common point corresponding to the temperature of the acetylene flame. In other words, if the spectrum of the acetylene flame were identical throughout with that of the carbon rod at the same temperature, the isotherm of the spectrum of the rod at that temperature would be a horizontal line. It is obvious, however, that if the wave-lengths of the middle of the spectrum should continue to increase faster than the red and the violet, a condition would presently be attained in which the ordinate of the isotherm would be greater in the yellow or green than at either end of the spectrum. We see indications of the approach of this condition in the diagram of isochroms (Fig. 9) from which it is evident that the curves for .65 μ and .60 μ would cut each other and would cut the curve for .70 μ at some temperature not far above 1400°, whereas the isochroms for the shorter wave-lengths would not be likely to cut the curves for the red until some much higher temperature had been reached. The curves in Fig. 10 show the nature of this unexpected development of the spectrum in a somewhat different aspect. It will be seen from this figure that the growth in the extreme red so far lags behind that the regions of the full red and this in turn behind that of the orange and this in turn behind that of the wave-length .6 μ, that at 1400° the

isotherm instead of being concave as seen throughout, actually becomes convex. I have indicated by means of lighter lines *a*, *b* and *c* the form of curve which might have been expected had the type of isotherm which exists at lower temperatures been maintained.

Above 1400° it was found impossible to obtain consistent readings on account of the rapid disintegration of the carbon rods, but I was able to satisfy myself after repeated trials that at temperatures not far above 1500° this change in the character of the isotherms had progressed to the point at which the yellow regions of the spectrum possess an ordinate greater than that of the extreme red or violet. At a temperature of about 300° below that of the acetylene flame the spectrum of the carbon rod was relatively weaker in the red, stronger in the yellow and weaker again in the shorter wave-lengths than the spectrum of the acetylene flame. There is no reason to suppose that had it been possible to heat the rods to the temperature of the flame itself the law of increase of intensity for the various wave-lengths would have undergone such radical modification to bring the two spectra at that temperature into identity.

Spectro-Photometric Measurements upon Rods with Treated Surfaces.

In order to compare the radiation of rods of black surface with those the surface of which have acquired a gray coating by treatment in hydrocarbon vapor, rods were mounted in the usual manner and after the exhaustion of the air from the metal box, gasoline vapor was allowed to enter until the atmosphere surrounding the rod was saturated. The rod was then brought several times to a high state of incandescence for a few seconds at a time, by which means the entire surface became coated with a gray deposit of carbon similar to that obtained by the treatment of incandescent lamp filaments. The metal box was then again pumped out and spectro-photometric measurements similar to those already described were made upon the radiation from the treated surface. It was thought that as the result of this treatment the carbon rods would stand a more prolonged exposure at high temperatures and thus it might be possible to extend the measurements beyond the points reached with the rods of black surface. This was found to be the case. As has already been indicated in a previous paragraph, the indications of a thermo

junction at these high temperatures was subject to serious suspicion. I was obliged to content myself therefore with estimations of the temperature based upon the difference of potential between the terminals of the rod. Fortunately the relation between the electromotive force and the temperature up to 1400° was of such a character that but little error was to be feared in extrapolating. The relation between electromotive force in volts and temperature is shown in Fig. 11. From this curve temperatures above 1400° were determined.

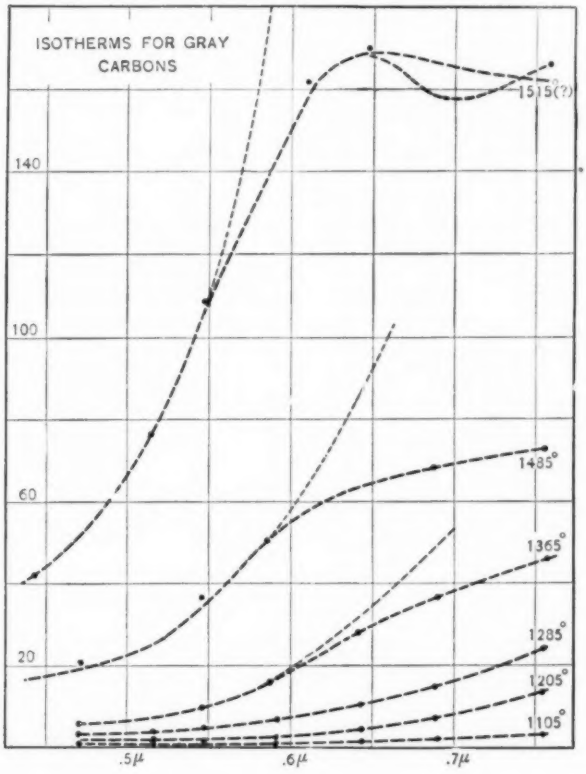

Fig. 11.

The work upon treated carbons was confined chiefly to high temperatures; a sufficient number of readings within the range already explored with the untreated carbons being taken to show that the distribution of intensities at the lower temperatures did not

differ materially from that in the spectrum of the former. The set of isotherms shown in Fig. 12, the data for which are given in Table III., will suffice to indicate the general character of the results. It

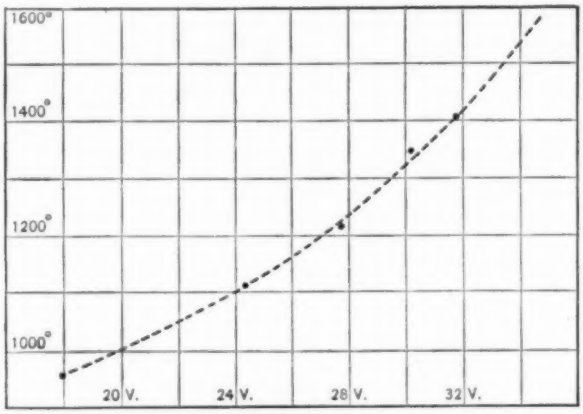

Fig. 12.

will be seen that in this case as in that of the untreated carbon the convexity of the curve between .6μ and the red end of the spectrum is well marked at 1365°; and that at 1515° there was a well pronounced maximum at about 65μ. The greater stability of the

TABLE III.

Relative intensities of various regions in the spectrum of gray (treated) carbon, the brightness of the spectrum of the acetylene flame being taken as unity throughout and the intensity of the region .6μ in the spectrum of carbon being taken as equal in brightness to that of corresponding region in the flame spectrum when the temperature of the rod is 1000° C.

	Value of $\frac{\text{Intensity of Carbon}}{\text{Intensity of } C_2H_2}$ at Temperatures Given Below.					
λ	1515°(?)	1485°	1365°	1285°	1205°	1105°
.760μ	166.0	72.7	45.5	23.8	13.3	2.4
.690μ	158.0	67.0	36.3	15.6	6.0	1.7
.647μ	170.0	52.5	28.3	10.7	3.6	1.5
.613μ	161.8					
.589μ	140.0	36.6	17.2	6.9	2.8	1.3
.548μ	108.8		10.2	5.4		0.7
.518μ	76.6			4.6	2.3	0.5
.470μ		20.5	5.6	3.5		0.3
.440μ	41.3					0.2

treated carbon made it possible to obtain consistent measurements on a number of rods at temperatures above 1500° and to establish beyond doubt the form of the curves.

It is obvious that for the study of the spectrum of incandescent carbon at this and higher temperatures the conditions would be much more favorable in the case of the incandescent lamp than with rods mounted in a large vacuum chamber like that used in the present investigation. Lamp filaments in the process of manufacture are brought by thorough carbonization into a condition to withstand permanently much higher temperatures than the rods at my disposal were capable of doing. There is as yet it is true no direct means of determining the temperature of the lamp filament; but the curve for the relation of electromotive force of temperature (Fig. 11) is of such a character as to lead us to expect that comparisons of the spectra of incandescent lamps, in which electromotive forces were used as a criterion of the degree of incandescence would at least enable us to confirm the existence of the remarkable phenomenon brought out by the present experiments and to extend observations of it to still higher temperatures.

Mr. Ernest Blaker has since the completion of the measurements described in this paper made comparisons of the spectrum of incandescent lamps with treated filaments and of lamps the filaments of which before exhaustion had been coated with lamp black with the spectrum of the acetylene flame. The results of his measurements confirm very completely those which I have described in this paper and contribute important evidence in favor of the existence of this anomaly from the law of distribution of intensities in the spectrum of glowing carbon.

Theoretical Aspects of the Foregoing Data.

The efforts of students of radiation have of late years been directed particularly to the testing of the various formulæ by means of which the mathematical physicists have attempted to express the intensity of radiation as a function of wave-length and temperature. The equation reached from quite different points of view by Wien[1] and Planck,[2]

[1] Wien, Wiedemann's Annalen, 58, p. 662, 1896.
[2] Planck, Drude's Annalen, 1, p. 69, 1900.

$$I = c_1 \lambda^{-5} e^{-\frac{c_2}{\lambda T}},$$

in particular, has been the subject of exhaustive discussion and of experimental tests. To this end Paschen[1] determined with the bolometer the distribution of energy in the infra-red spectra of various bodies from 15° C. to 1300°. The materials thus subjected to measurement were oxide of copper, platinum, lamp black, and graphitic carbon. The range of wave-lengths explored extended from 9.2 μ to 0.7 μ. Lummer and Pringsheim[2] and still more recently Mendenhall and Saunders[3] have made similar determinations with the ideal black body; and Lummer and Jahnke[4] have compared the infra-red spectrum of such a body with that of platinum. Wanner[5] working with Paschen made careful spectro-photo metric measurements of the visible radiation from the ideal black body. To test the applicability of the Wien-Planck formula to these measurements of the radiation the equation is given the form,

$$\log I = \gamma_1 - \gamma_2 \frac{1}{T},$$

in which

$$\gamma_1 = \log (c_1 \lambda^{-5}),$$

$$\gamma_2 = \frac{c_2}{\lambda} \log e.$$

The isochromatic curves are then plotted with the logarithm of the intensities as ordinates and the reciprocal of the absolute temperature as abscissæ. The agreement of he equattion with the observations is found in the fact that isochroms thus plotted, at least as far as the work of Paschen and Wanner is concerned, always take the form of straight lines and that the quantity, c_2, computed for various wave-lengths is found to be a constant. Lummer and Pringsheim on the contrary find in the discussion of their measurements that the

[1] Paschen, Wiedemann's Annalen, 58, p. 445, 1896; also, Wiedemann's Annalen, Vol. 60, p. 662, 1897.

[2] Lummer and Pringsheim, Deutschen Phys. Gesellschaft, I., p. 23; II., p. 163, 1900.

[3] Mendenhall and Saunders, Astrophysical Journal, Vol. 13, p. 25, 1901.

[4] Lummer and Jahnke, Drude's Annalen, Vol. 3, p. 283, 1900.

[5] Wanner, Drude's Annalen, 2, p. 141, 1900.

constant c_2 increases steadily with a wave-length from 13500 at 1.2μ to 16500 at 5μ and 18500 at 8.3μ. The value of c_2 computed by measurements from Beckman at wave length 24 was found to be 24250. Lummer and Pringsheim find moreover that the logarithmic isochroms, especially when extended to higher temperatures, are not straight lines, but show a slight convexity towards the $\frac{1}{T}$ axis.

Exception has also been taken to the Wien-Planck formula on the ground that it gives for infinite temperatures a finite limit to the value of the intensity; a result which Rayleigh[1] in a recent paper has characterized as *physically improbable*.

Rayleigh proposes form

$$I = c_1 T \lambda^{-4} e^{-\frac{c_2}{\lambda T}}$$

but Lummer and Pringsheim find that this likewise fails to properly express their experimental results. Lummer and Jahnke propose in view of these discrepancies to give the equation the general form

$$I = c T^5 (\lambda T)^{-\mu} e^{-\frac{c}{(\lambda T)^s}}$$

an expression which coincides with Wien's formula for $\mu = 5$ and with Rayleigh's for $\mu = 4$. They find the measurements of Lummer and Pringsheim satisfied when μ lies between 4.5 and 5 and $\checkmark$ lies between .9 and 1.0. If we accept the value $\mu = 5$ and $\checkmark = 0.9$ this equation always leads to a finite value of intensity for infinite temperature. All other values of these quantities give infinity as the limit of intensity.

Whether logarithmic isochroms or the value of the quantity c_2, computed from measurements upon carbon rods, would aid in deciding between the various equations under discussion is a question. The data given in this paper would not lead us to class the carbon rod studied as *black bodies*. The emissive power of various forms of carbon is well known to be smaller than that of the ideal black body and there is no reason to suppose that it is independent of the temperature. The relative lagging behind of the intensities in the

[1] Philosophical Mag., Vol. 49, p. 539, 1900.

red might perhaps be taken as an indication of a tendency to approach the finite maximum demanded by the Wien-Planck formula; but the isochrom for .76 shows that the effect if it exists must be looked for at some much higher temperature than that covered by these measurements. In spite of these doubts as to the applicability as to the measurements on carbon rods to the problem of the law of radiation of the ideal black body, I have plotted the various iso-

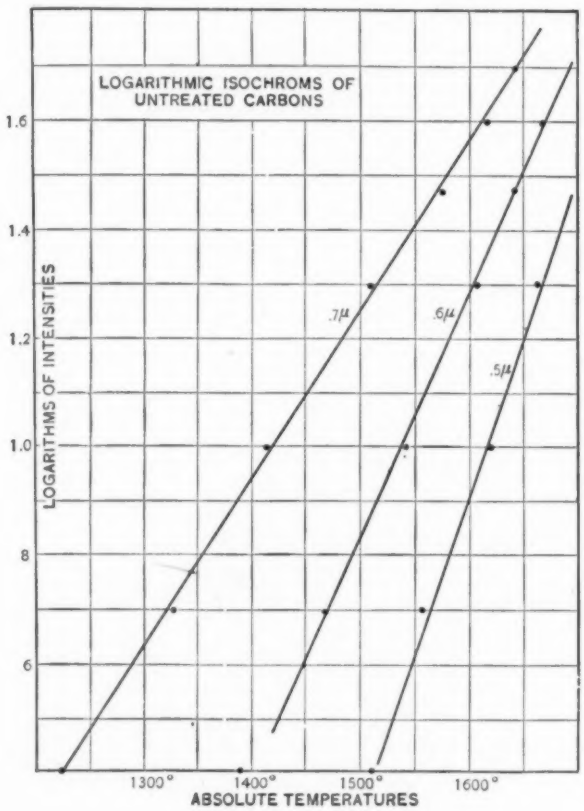

Fig. 13.

chroms obtained in the course of this investigation in logarithmic form; absolute temperatures being taken as abscissæ and the logarithm of the intensity as ordinates. As will be seen from Fig. 13, in which three curves from Fig. 9 are reproduced, these loga-

rithmic isochroms are straight lines. The range of temperatures is doubtless too small to bring out the curvature found by Lummer and Pringsheim; but they show clearly the change of direction with the wave-length mentioned by them on page 222 of their paper before the German Physical Society.[1]

In carrying on portions of the work described in these papers I have had valuable help from Drs. C. H. Sharp and Leopold Kann and from Mr. L. W. Hartman, to all of whom I desire to express my obligations and to extend my hearty thanks.

PHYSICAL LABORATORY OF CORNELL UNIVERSITY, April 24, 1901.

[1] Lummer and Pringsheim, Verhandl. d. Deutschen. physikal. Gesellsch., 1899, p. 222.

ON THE PERIOD OF A ROD VIBRATING IN A LIQUID.

By Mary I. Northway and A. Stanley Mackenzie.

As early as 1786 Buat announced in his "Principes d'Hydraulique" that the period of a pendulum was greater when vibrating in a fluid than when vibrating in a vacuum, not only because of the loss in weight due to buoyancy but also because of the mass of fluid which must be considered as participating in the motion of the pendulum. The latter is loaded by, or drags with it, a certain amount of the fluid. Buat determined this added mass for the case of a sphere and for bodies of other forms. Little attention was paid to this work until Bessel[1] about forty years later, in determining the length of the seconds pendulum, concluded from theoretical reasons that it was necessary to take into account the inertia of the air as well as its weight. When the dimensions of the bob were small in comparison with the length of the suspending wire, Bessel represented the apparent increase in the weight of the pendulum by a constant, k, times the mass of the displaced fluid. For a sphere 2 in. in diameter vibrating in water he found for k the value 0.9459; later he changed this to 0.956.

Sabine[2] tried to find the effect of air on a pendulum by making it vibrate in air and then in a vacuum. He found that the old correction for buoyancy should be multiplied by a factor m, whose value was 1.655, in order to obtain the total correction.

In 1832 Baily[3] published the results of some similar experiments. Four spheres of about 1.5 in. in diameter gave for m the value 1.84; and three spheres of about 2 in. in diameter gave 1.748. For spheres of about the same size Bessel had found the value 1.956. For small cylinders Baily found the value of m to increase regularly with decrease of diameter, but according to no apparent law.

[1] Abh. d. Akad. d. Wiss., Berlin, 1826; Coll. de Mem. Soc. Fr. de Phys., **5**, 1.

[2] Phil. Trans., 1829; Coll. de Mem. Soc. Fr. de Phys., **5**, 134.

[3] Phil. Trans., 1832, p. 399; Coll. de Mem. Soc. Fr. de Phys., **5**, 185.

Poisson[1] determined analytically the value of m for spheres and found it to be 1.5 ; which agrees very well with the results of Buat's experiments made fifty years earlier.

In 1843 Stokes[2] found from theoretical considerations that for a long cylinder oscillating in a direction perpendicular to its length, the effect of the inertia of the fluid was the same as if a mass equal to that of the displaced fluid were distributed along the axis of the cylinder. This agrees very well with the experimental results obtained by Baily for a long tube 1.5 in. in diameter.

Up to this time no law was known connecting the variation of m with the dimensions of the pendulum, and the only property of the fluid taken into consideration was its density. In 1848 Stokes[3] treated the problem of a body oscillating in an infinite liquid medium. He dealt with the cases of a disk rotating about its axis, of a sphere oscillating in the direction of a diameter, and of an infinitely long cylinder vibrating in a direction perpendicular to its length. He considered the effect of the viscosity as well as of the density of the fluid. This paper is of especial interest here because the experiments about to be described approximate more nearly to this problem than to any other that has been treated mathematically. The main results of this important piece of work will be referred to later.

In 1880 Montigny[4] published an account of some experiments which he had made in 1859 in order to find the effect of the density and of the compressibility of various liquids on the pitch of bells. (He intended these experiments to be preliminary only and to repeat them with electrically driven forks. He evidently did not do so, although in a second paper[5] he describes his method of maintaining forks in vibration in a liquid, and states that the pitch is lower the denser the liquid.) He found the pitch of the bell first in air, and then when submerged mouth upward in the liquid; also when in air and filled to the brim with the liquid; and finally when

[1] Mem. de l'Acad. de Paris, 11, 1831.

[2] Trans. Camb. Phil. Soc., 8, 105, 1843; B. A. Report, 1848.

[3] Trans. Camb. Phil. Soc., 9, pt. 2, p. 35; Coll. de Mem. Soc. Fr. de Phys., 5, 277.

[4] Bull. de l'Acad. de Belg., [2], 50, 159, 1880.

[5] Bull. de l'Acad. de Belg., [2], 50, 300, 1880.

empty and immersed to the brim in the liquid. The pitch was the same for the last two cases, and the lowering of pitch was less than when the bell was entirely submerged; from this he concluded that the lowering was due to that part of the liquid which was in contact with the bell. The lowering was found to depend also on the shape and material of the bell. The pitch was determined by comparing the note by ear with that of a monochord. He found that the lowering of pitch was greater the denser the liquid and the graver the pitch. He gives the following intervals of lowering[1] for four of the bells used.

	Sol_4	Ut_5	Sol_5	Ut_6
When filled with water . .	1.204	1.125	1.127	1.118
When submerged in water .	1.411	1.327	1.286	1.261

Auerbach,[2] in a paper written in 1878, attempted to deduce a simple theoretical relation between the pitch of a tuning fork in air and that in a liquid as a consequence of the different way in which kinetic energy is dissipated in the two cases. The pitch, he contends, always depends on the square root of the coefficient of elasticity, and this can have two entirely different values according as the vibration takes place isentropically or isothermally. The former he considers to be the case for air and the latter for liquids. He then takes the coefficients of elasticity in the two cases to be as the ratio of the specific heats, 1.4 : 1, and the interval of lowering of pitch to be as $\sqrt{1.4} : 1$; that is, as 1.18 : 1. He tried four forks in air and in water, observing the pitch by ear, and obtained the following intervals:

Frequency of fork in air,	132	264	396	528
Interval of lowering,	1.11	1.12	1.13	1.15

He concluded from these experiments that the ideal interval, 1.18, would be approached more and more nearly the higher the pitch. He adds that experiments with other liquids show that the density and viscosity of the liquid have no influence on the lowering of pitch.

[1] The ratio of the pitch in air to that in the liquid is commonly called the interval of lowering.

[2] Wied. Ann., 3, 157, 1878.

Koláček,[1] referring to this work by Auerbach, suggests that an explanation of this lowering of pitch can be given from simple mechanical principles, and that the effect of the liquid is mainly to load the fork by a certain quantity which is the product of the density, ρ, of the liquid and a factor, c, depending on the size and shape of the vibrating body. Accordingly he finds the interval of lowering to be $\frac{T'}{T} = \sqrt{1 + c\rho}$. Taking the mean, 1.13, of the intervals found by Auerbach, he determined the value for c to be 0.277. Using this value of c it follows that Auerbach's forks should give $\frac{T'}{T}$ equal to 2.18 when made to vibrate in mercury, and 1.21 when in sulphuric acid. On plunging a 435 fork into various liquids and noting its changed pitch by an ear comparison with a monochord, Koláček found the value of $\frac{T'}{T}$ to be 2.1 for mercury, 1.21 for sulphuric acid, and about the same for ether and alcohol as for water. He proceeds to treat analytically the simple cases of the radial dilatational and linear translational oscillations of a sphere in a liquid, and finds expressions for $\frac{T'}{T}$ in the two cases. When applied to a solid iron sphere vibrating in water these expressions give for $\frac{T'}{T}$ the values 1.03 and 1.17 respectively. Neither of these cases is that of a tuning fork, but they give limiting values for it and the mean of the two agrees roughly with Auerbach's results. In a later paper Koláček[2] gives a more rigorous hydrodynamical discussion of the general problem of the vibration of a solid in a liquid.

In 1882 Auerbach[3] experimented with glass cylinders filled with liquid. Assuming that the effect of the liquid was to load the cylinder with a mass, c, depending on the nature of the liquid and on the shape and dimensions of the cylinder, he deduced the equation, $\frac{T'}{T} = \sqrt{\frac{m+c}{m}}$, for the lowering of pitch, where m is the mass of the cylinder. He obtained the following results: (1) the interval was

[1] Wied. Ann., 7, 23, 1879.
[2] Sitzb. math.-naturw. Cl. Wien, 87, Abth. 2, 1147, 1883.
[3] Wied. Ann., 17, 964, 1882.

smaller the higher the pitch ; (2) the interval was independent of the height of the cylinder ; (3) the interval was greater in proportion to the narrowness of the cylinder ; (4) using different liquids whose densities varied from 0.729 to 1.364, and in addition mercury, he found that as a first approximation the specific lowering of pitch (the ratio of the lowering for any liquid to that for water) depended on the density only and increased with it, but more slowly, and as a second approximation that the specific lowering increased as the compressibility of the liquid decreased. He determined the pitch by Kolácek's method.

Miss L. R. Laird,[1] working in this laboratory, made some experiments on the change in the period of a pianoforte wire 0.446 mm. in diameter and 107 cm. long, vibrating in the following liquids : water, solutions of potassium carbonate, and mercury. The wire was made to vibrate by means of an electro-magnet and its record was taken on a revolving drum along with that of a standard tuning-fork. The results for the lowering of pitch agreed very well with the values obtained analytically by Stokes in his 1848 paper for the case of an infinite cylinder. The change of pitch gave the following intervals : 1.06 for water ; 1.13 for potassium carbonate of density 1.47 ; 1.095 for potassium carbonate of density 1.22 ; 1.72 for mercury. In the case of water the observed values agreed still more closely with numbers calculated on the simple supposition that the wire was loaded with the mass of the displaced liquid, which was the conclusion arrived at by Stokes in his 1843 paper. Some experiments on a similar wire 0.933 mm. in diameter and 37 cm. long gave the interval 1.065 for water, 1.18 for oil of density about 0.9, and 1.12 for a sodium nitrate solution of density 1.38. It will be noticed that the lowering for water was the same for both wires.

The following experiments were made in continuation of those of Miss Laird, and were intended to separate, if possible, the effects of the density and the viscosity of the liquid. The change in pitch of a tuning fork is of most direct interest, and, as the nearest simple approach to this, a rod clamped at one end was studied. As has been stated, this problem is a very rough approximation to the case of the infinite cylinder discussed by Stokes. The present problem

[1] PHYS. REV., 7, 102, 1898.

differs from his in that the rod is of a different cross section and has edges, which greatly changes the stream lines; that it has not the same amplitude at all points, and is of finite length. Nevertheless, a consideration of Stokes' work will be of great service, and will throw at least some light upon the way in which the constants of the liquid must enter.

If we suppose that a mass m has a simple oscillatory motion as a whole in the x direction, and is in a vacuum, its equation of motion is

$$m\frac{d^2x}{dt^2} + px = 0; \tag{1}$$

and its period

$$T = 2\pi\sqrt{\frac{m}{p}}. \tag{2}$$

Stokes found that the effect of a liquid medium is to introduce two terms into the above equation of motion. The first of these is of the nature of an acceleration, the second of a velocity. The former is equivalent to an increase of the mass of the vibrating body by a certain fraction of the mass of the displaced fluid; m becomes $m + km'$ where m' is the mass of the displaced fluid. This is the same as saying with Buat that the body drags along with it a definite quantity of the liquid. The quantity k will evidently be determined by the form of the stream lines and must accordingly be a function of the shape and size of the cross-section of the vibrating body, of the density of the liquid, and of the coherence of its separate layers, or of its viscosity; and this Stokes found to be the case. The second of the terms which he found to enter into the equation of motion is a retarding force proportional to the velocity. The form of this term can be assigned from general considerations; for the energy absorbed by the friction will be determined by the amount of the fluid displaced, and will depend not only on the velocity but also on the number of times its direction is changed. Stokes found it to be of the form $k'm'n$, where n is the pitch of the body in vacuum, and k' a quantity depending on the dimensions of the body and on the density and viscosity of the liquid. For the infinitely long cylinder he found, when $\frac{2}{a}\sqrt{\frac{1}{\pi n}\cdot\frac{\mu}{\rho}}$ was small,

$$(3) \qquad k = 1 + \frac{2}{a}\sqrt{\frac{1}{\pi n}\cdot\frac{\mu}{\rho}}, \quad \text{and} \quad k' = \frac{2}{a}\sqrt{\frac{1}{\pi n}\cdot\frac{\mu}{\rho}},$$

and hence $k = 1 + k'$ where a is the radius of the cylinder.

The quantity $\sqrt{\frac{1}{\pi n}\cdot\frac{\mu}{\rho}}$ enters also into the values of k and k' found for the disk and the sphere.

Our equation of motion (1) can now be written

$$(4) \qquad (m + km')\frac{d^2x}{dt^2} + k'm'n\frac{dx}{dt} + px = 0.$$

The well-known solution of this equation gives for the period T' the value

$$T' = 2\pi\left[\frac{p}{m + km'} - \tfrac{1}{4}\left(\frac{k'm'n}{m + km'}\right)^2\right]^{-\frac{1}{2}}$$

$$= 2\pi\left(\frac{m + km'}{p}\right)^{\frac{1}{2}}\left[1 + \tfrac{1}{8}\frac{k'^2m'^2n^2}{p(m + km')}\right]$$

approximately, assuming k'^2 to be small. This can be put in the form

$$T' = 2\pi\left(\frac{m}{p}\right)^{\frac{1}{2}}\left[1 + k\frac{\rho}{\Delta}\right]^{\frac{1}{2}}\left\{1 + \tfrac{1}{8}k'^2\left(\frac{\rho}{\Delta}\right)^2\frac{1}{1 + k\frac{\rho}{\Delta}}\right\}$$

where Δ is the density of the body. Using equation (2) this becomes

$$\frac{T'}{T} = \left(1 + k\frac{\rho}{\Delta}\right)^{\frac{1}{2}}\left\{1 + \tfrac{1}{8}k'^2\left(\frac{\rho}{\Delta}\right)^2\frac{1}{1 + k\frac{\rho}{\Delta}}\right\}$$

If k' is at all small we may neglect the second factor, especially for liquids of small density; in this case the interval of lowering of the pitch becomes

$$(5) \qquad \frac{T'}{T} = \sqrt{1 + k\frac{\rho}{\Delta}}.$$

This is equivalent to neglecting in our equation of motion the term which depends on the velocity. The following experiments were

made to test this equation and to find the form and value of the quantity k. One could approximate more closely to Stokes' theoretical problem than was done in the present investigation by using rods of circular cross-section; but the mechanical difficulties of maintaining such rods in vibration in a plane are too great, and those used had a rectangular cross-section, and so corresponded to one-half of a tuning-fork. Although in the theoretical discussion T is the period in a vacuum, it is a sufficiently good approximation for this investigation to take it as the period in air.

The rod to be experimented on was held in a heavy clamp screwed to the inside of a wooden box, which held, when required, the liquid under consideration. The rod was maintained in vibration by an electro-magnet which could be suspended at any height from a movable bar across the top of the box. The circuit was completed through the rod and a piece of platinum soldered to its free end dipping into a mercury cup fitted with a plunger and projecting up from the bottom of the box. When the rod was of brass a thin piece of soft iron 9 mm. wide and 32 mm. long was soldered to it just beneath the electro-magnet. The vibrations were recorded on a revolving drum, upon which a marker also recorded seconds as given by a standard clock. The pitch of a given length of the rod used was found first with the vibrations taking place in air; then without disturbing anything the box was filled with the liquid under consideration, and the lowering of pitch calculated. The box was large in order that the nearness of its walls might not influence appreciably the pitch of the rod. Three boxes were used, as follows:

	With Water.	With Oil.	With Sodium Nitrate Solution.
Internal length.	57 cm.	55 cm.	55 cm.
" breadth	26 "	21 "	25 "
" depth	30 "	22 "	22 "

To show that the size of the box is of great importance it may be stated that the pitch of the rod A (described below) in water was increased nearly five per cent. when it was changed from the large box to one whose dimensions were $52 \times 4.5 \times 4.5$ cm. In order to

prevent as far as possible the absorption of energy by the box, etc., the clamp was made of a mass of metal weighing fifteen pounds; the whole apparatus was placed on a stone pier; and the box, the cross-bar supporting the electro-magnet, and the projecting end of the rod, if any, were heavily weighted down with masses of iron. Unless this were done the pitch was sensibly affected.

The duration of contact of the platinum wire with the mercury, the distance of the electro-magnet and piece of soft iron from the clamp, the shape of the electro-magnet and its height above the rod, and the strength of the driving current were all found to affect the pitch of the rod. The time of contact used was always the shortest that would maintain the rod in vibration. The distance of the electro-magnet from the clamp affected the pitch in the following way: the shorter the distance the greater was the interval of lowering, no matter what was the vibrating length of the rod. Table I. gives a specimen of the readings taken for the purpose of studying this question; they were made on the rod *A* with half of its length (21.1 cm.) vibrating.

TABLE I.

Distance of Electro-magnet from Clamp.	Frequency in Air.	Frequency in Water.	Height of Electro-magnet above Soft Iron.	Interval of Lowering.	Average Interval.
2.1 cm.	44.67	38.45	3 mm.	1.162	1.162
" "	44.63	38.40	3 "	1.162	
" "	44.62	38.37	3 "	1.163	
11.0 "	38.89	33.63	8 "	1.153	1.154
" "	39.09	33.72	8 "	1.156	
" "	39.09	33.91	8 "	1.153	
17.5 "	36.92	32.21	8 "	1.146	1.147
" "	36.95	32.22	8 "	1.147	
19.5 "	36.64	32.09	8 "	1.142	1.142
" "	36.63	32.08	8 "	1.142	

The final average intervals, found as above, for various lengths of the rod were:

Average distance of electro-magnet,	6 cm.	11 cm.	19 cm.
Average interval,	1.160	1.153	1.148

The interval, therefore, diminishes as the driving force is placed nearer to the free end of the rod. In order to get results undisturbed by this possible source of variation, the electro-magnet was

placed at a constant distance from the clamp throughout the whole series of experiments, the piece of soft iron being taken off the rod and soldered on in a new place whenever the vibrating length of the rod was changed. The distance from the clamp chosen was such that with the shortest length of rod used the soft iron was at the tip end. For the rod *A* this length was 10.55 cm. and for the other rods it was 21.1 cm. A set of observations was taken to find also the effect of the height of the electro-magnet above the rod. The electro-magnet first used was of the *U* shape. Its cores did not project beyond the bobbins, whose ends were each about 2 sq. in. in area. The effect of the height was not very noticeable for air, but for water a change of height from 7 to 23 mm. made the rod vibrate more slowly by more than 1 per cent. There was no doubt that with this electro-magnet near the rod, the coils acted partly as a wall, and they were replaced by a single coil whose core projected 15 mm. and was 6 mm. in diameter. With this electro-magnet it was found that a diminution of height increased the pitch, no matter how far the electro-magnet was from the clamp, but in no case by so much as 1 per cent. and usually by much less. It was evident, therefore, that the height must be kept constant throughout the experiments; but as the height of the electro-magnet was increased, the current also had to be increased, and it seemed desirable to keep the current also constant. The current strength chosen was the smallest that would drive the shortest length of rod used, and the height chosen was the greatest possible for that current and length.

Five rods were used during the investigation. Four of them, called *A*, *B*, *C* and *D*, were of brass, and one, *E*, of steel. *B* had approximately twice the thickness of *A* and the same width; *C* twice the width and the same thickness; and *D* twice the width and twice the thickness. *E* had approximately the same dimensions as *A*. The dimensions and densities of the rods were:

	A	*B*	*C*	*D*	*E*
Width. . . .	0.94 cm.	0.97 cm.	1.91 cm.	1.95 cm.	0.97 cm.
Thickness. .	0.32 "	0.65 "	0.34 "	0.67 "	0.33 "
Density. . .	8.56	8.56	8.70	8.56	7.79

The liquids used were water, cotton-seed oil, and a solution of sodium nitrate saturated at 15° C. The first two had approximately the same density but very different viscosities, while the first and last had approximately the same viscosity but quite different densities. The viscosities of the oil and the solution of sodium nitrate were found by the method used by Gartenmeister.[1] The temperature throughout was kept as nearly as possible at 17°.6 C. At this temperature the liquids have the following constants :

	Water.	Cotton-Seed Oil.	Sodium Nitrate Solution.
Density, ρ	1.000	0.921	1.355
Viscosity, μ	0.011	0.781	0.030

The tables which follow give the results obtained. All of the separate readings are not given, but the average frequency in air and in the liquid for each length, and the corresponding interval of lowering. In no case were less than two distinct sets of observations made on each length of the rod used. Ordinarily 1,000 vibrations for each observation were counted on the drum and the corresponding number of clock seconds, and the pitch calculated therefrom.

TABLE II.

Rod A. Dimensions, 42.2 × 0.94 × 0.32 *cm. In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	10.33	4	9.04	3	1.148
¾ "	18.16	2	15.78	2	1.151
⅝ "	25.58	2	22.27	2	1.149
½ "	39.10	2	33.95	2	1.152
⅜ "	66.73	2	58.09	3	1.149
¼ "	136.44	4	119.45	6	1.143
Average Lowering.					1.149

From this table it is seen that for a range of about 3½ octaves the lowering is independent of the pitch within the limits of accuracy

[1] Zeitschr. f. physikal. Chem., 6, 524, 1890.

attained. The same deduction can be made from the numbers in each of the other tables. The lowering here is a little less than a minor whole tone plus a small semitone $\left(\frac{10}{9} \times \frac{25}{24} = 1.157\right)$.

TABLE III.

Rod E. Dimensions, 42.2 × 0.97 × 0.33 *cm. In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
¾ length.	24.86	2	21.48	2	1.157
⅝ "	35.62	2	30.71	2	1.159
½ "	53.54	2	46.20	2	1.158
Average Interval.					1.158

The lowering here again is independent of the pitch. The interval is exactly a minor whole tone plus a small semitone, and differs by less than 1 per cent. from the interval for the brass rod of approximately the same dimensions. It seems safe to conclude that no great importance is to be attached to the materials used for the rods, at least for such materials as are ordinarily used for tuning forks, bells and kindred musical instruments.

TABLE IV.

Rod B. Dimensions, 42.0 × 0.97 × 0.65 *cm. In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	19.80	2	18.29	2	1.083
¾ "	34.56	2	31.65	2	1.094
⅝ "	47.50	2	43.60	2	1.089
½ "	72.14	3	66.51	3	1.085
Average Interval.					1.088

The lowering is nearly a semitone plus a comma, or a great limma ($\frac{16}{15} \times \frac{81}{80} = 1.080$), and is not far from being one-half of that for rod *A* of one-half the thickness.

TABLE V.

Rod C. Dimensions, 42.2 × 1.91 × 0.34 *cm. In water.*

Vibrating Length.	Frequency in air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	10.12	2	7.95	2	1.273
¾ "	17.81	2	14.01	2	1.274
⅝ "	25.05	2	19.78	2	1.267
½ "	37.86	5	30.32	5	1.244
Average Interval.					1.265

The lowering is exactly a major third plus a comma ($\frac{5}{4} \times \frac{81}{80} = 1.265$) and is nearly twice as much as that for rod *A* of one-half the width.

TABLE VI.

Rod D. Dimensions, 42.0 × 1.95 × 0.67 *cm. In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	19.89	3	17.19	3	1.157
¾ "	35.13	8	30.29	7	1.160
⅝ "	48.23	2	42.34	2	1.140
Average Interval.					1.152

The lowering here is a little less than a minor whole tone plus a small semitone, and is almost exactly the same as that for rod *A* of one-half the width and one-half the thickness, as we should expect from the results shown in Tables IV. and V.

GENERAL APPROXIMATE RESULTS.

1. The interval of lowering for a rod of given cross section is independent of the length.

2. The interval of lowering for a rod of given cross section is approximately the same for brass and steel and is probably independent of the material within the range of substances ordinarily used.

3. The interval of lowering for a rod of given width is approximately inversely proportional to the thickness.

4. The interval of lowering for a rod of given thickness is approximately directly proportional to the width. More definite relations will be given later.

The above experiments were then repeated with cotton-seed oil instead of water. This oil was used because it is inexpensive, and consequently can be used in large quantities, and has its coefficient of viscosity about seventy times that of water. With this liquid it was hoped that the importance of the viscosity in the lowering of pitch could be determined. Whereas in the case of water the contact breaker was under the liquid, it had to be outside for the oil and the nitrate solution. The platinum wire was replaced by a platinum-tipped thin steel wire which reached above the surface of the liquid where it was bent over and dipped into the mercury cup. No special difficulty was experienced in maintaining the rod in vibration in the viscous liquid. The results of these experiments were as follows:

TABLE VII.

In Cotton-seed Oil.

Rod.	Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Oil.	Number of Readings.	Interval of Lowering.	Av. Interval of Lowering.
A	Full length.	10.00	10	8.67	10	1.140	
"	¾ "	17.32	2	14.81	2	1.169	
"	⅝ "	24.57	2	20.91	2	1.175	
"	½ "	37.15	4	31.90	4	1.164	1.162
B	Full length.	20.26	3	18.42	4	1.105	
"	¾ "	36.11	3	32.22	3	1.121	
"	⅝ "	49.67	4	44.72	3	1.111	
"	½ "	65.18	3	59.65	3	1.093	1.108
C	Full length.	10.21	4	8.13	4	1.256	
"	¾ "	18.38	3	14.33	3	1.283	
"	⅝ "	25.76	3	19.90	3	1.294	
"	½ "	38.60	2	30.58	2	1.262	1.272
D	Full length.	19.67	3	16.89	3	1.166	
"	¾ "	35.30	2	29.54	5	1.195	
"	⅝ "	48.43	3	41.44	3	1.169	1.176

We shall discuss these numbers more carefully later, but we can say at once in a general way that the effect of viscosity is not very marked. In order to throw still more light on the effect of viscosity as compared with that of density, the four brass rods were made to vibrate in a solution of sodium nitrate, which had a viscosity of only 0.0299, and so not very much greater than that of

water, and whose density was 1.355, about a third as much again as that of water. These determinations resulted as follows:

TABLE VIII.

In Sodium Nitrate Solution.

Rod.	Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Nitrate Solution.	Number of Readings.	Interval of Lowering.	Av. Interval of Lowering.
A	Full length.	10.00	2	8.47	3	1.181	
"	¾ "	17.25	2	14.48	3	1.191	
"	⅝ "	24.70	2	20.46	3	1.207	
"	½ "	37.58	2	31.51	2	1.193	1.198
B	Full length.	20.35	2	18.10	2	1.123	
"	¾ "	35.75	2	31.84	2	1.123	
"	⅝ "	50.99	2	44.96	2	1.134	1.127
C	Full length.	10.21	2	7.61	2	1.343	
"	¾ "	18.51	2	13.76	2	1.347	
"	⅝ "	25.98	2	19.26	2	1.349	1.346
D	Full length.	19.97	2	16.57	2	1.206	
"	¾ "	35.30	3	29.51	3	1.196	
"	⅝ "	49.15	3	40.88	3	1.202	1.201

Comparing these numbers with those obtained for water it is seen that the lowering is increased very decidedly, and that the increase is not far from being directly proportional to the added density. It is evident therefore that the main factor in the lowering of pitch is the density of the medium, and that the effect of viscosity is relatively small.

Thus far the conclusions drawn from the investigation have been of an approximate and general nature; it remains to determine whether an expression for $\frac{T'}{T}$ can be found which will contain the results of every case investigated. Stokes' work can afford little direct help since he found the value of $\frac{T'}{T}$ to depend on n, as will be seen by reference to equation (3); whereas in the present investigation the interval of lowering is independent of the pitch within the limits of accuracy attained. Indirectly his work is of great service, and his results have been used already in the formation of the equation of motion (4). Since for the case he discussed, the value of k'

is considerably smaller than that of k, and since the results here obtained for oil show that the change of pitch due to friction is relatively small, it is allowable as a first approximation to neglect k', and consider the lowering of pitch to be expressed by equation (5),

$$\frac{T'}{T} = \sqrt{1 + k\frac{\rho}{\Delta}}.$$

The following table contains the values of k found from this equation.

TABLE IX.

Values for k.

Medium.	Vibrating Length.	Rod A.	Rod B.	Rod C.	Rod D.	Rod E.
Water.	Full length.	2.697	1.478	5.396	2.895	
	¾ "	2.776	1.682	5.418	2.954	2.641
	⅝ "	2.737	1.589	5.263	2.561	2.678
	½ "	2.796	1.515	4.761		2.660
Average.		2.752	1.566	5.210	2.803	2.660
Oil.	Full length.	2.785	2.054	5.454	3.342	
	¾ "	3.407	2.385	6.101	3.978	
	⅝ "	3.537	2.178	6.369	3.407	
	½ "	3.298	1.809	5.596		
Average.		3.257	2.107	5.880	3.576	
$NaNO_3$	Full length.	2.483	1.642	5.152	2.858	
	¾ "	2.632	1.643	5.220	2.707	
	⅝ "	2.873	1.799	5.255	2.797	
	½ "	2.662				
Average.		2.663	1.698	5.209	2.787	

The quantity k can be a function of the constants of the liquid and the dimensions of the rod. Stokes has shown that μ and ρ enter always as a ratio, and further that in the equation for the lowering of pitch they enter as $\sqrt{\frac{\mu}{\rho}}$; an attempt was accordingly made to express k as a function of $\sqrt{\frac{\mu}{\rho}}$ and the cross section of the rod (the length has been shown not to enter). This it has been found can be done in several ways of about the same degree of satisfactoriness. The following four equations give values for $\frac{T'}{T}$

with various forms of the quantity k; w denotes the width of the rod, and t the thickness.

(*a*) $$\frac{T'}{T} = \sqrt{1 + 2.63 \frac{w^2}{t^2}\left(1 + \tfrac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\Delta}}.$$

(*b*) $$\frac{T'}{T} = \sqrt{1 + 1.12 \frac{w}{t^2}\left(1 + \tfrac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\Delta}}.$$

(*c*) $$\frac{T'}{T} = \sqrt{1 + \frac{w}{t^2}\left(1 + \tfrac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\Delta}}.$$

(*d*) $$\frac{T'}{T} = \sqrt{1 + 0.91 \frac{w}{t}\left(1 + \tfrac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\Delta}}.$$

The last of these equations is preferred on account of its simplicity, but its concordance with the observed values is not so good as is the case with the first, where the greatest deviation is 1 part in 110. The observed average values for $\frac{T'}{T}$ and those calculated from equation (*d*) are given in Table X., along with their differences.

TABLE X.

Rod.	Medium.	Water.	Oil.	$NaNO_3$.
A	Obs.	1.149	1.162	1.198
	Cal.	1.150	1.164	1.200
	Diff	+ .001	+ .002	+ .002
B	Obs.	1.088	1.108	1.127
	Cal.	1.078	1.086	1.106
	Diff.	− .010	− .022	− .021
C	Obs.	1.265	1.272	1.346
	Cal.	1.271	1.295	1.358
	Diff.	+ .006	+ .023	+ .012
D	Obs.	1.152	1.176	1.201
	Cal.	1.148	1.163	1.198
	Diff.	− .004	− .013	− .003
E	Obs.	1.158		
	Cal.	1.162		
	Diff.	+ .004		

It will be seen that the agreement is as close as that of the separate values of $\frac{T'}{T}$ in any of the tables II. to VIII.

The above equations (a), (b), (c) and (d) were found from a close consideration of Stokes' work. It is worthy of remark, however, that an entirely empirical formula can be found which will give an equally good agreement with the results. The following table contains the observed values of $\frac{T'}{T}$, and those calculated from the equation

(e) $$\frac{T'}{T} = 1 + 0.45 \left[\frac{a^2 \mu}{l}\right] + 0.03\,\mu.$$

TABLE XI.

Rod.	Medium.	Water.	Oil.	$NaNO_3$.
	Obs.	1.149	1.162	1.198
A	Cal.	1.157	1.168	1.212
	Diff.	+ .008	+ .006	+ .014
	Obs.	1.088	1.108	1.127
B	Cal.	1.079	1.096	1.107
	Diff.	− .009	− .012	− .020
	Obs.	1.265	1.272	1.346
C	Cal.	1.273	1.275	1.368
	Diff.	+ .008	+ .003	+ .022
	Obs.	1.152	1.176	1.201
D	Cal.	1.180	1.196	1.254
	Diff.	+ .028	+ .020	+ .053
	Obs.	1.158		
E	Cal.	1.155		
	Diff.	− .003		

In the work which has been done on this subject there is only one case where sufficient details are given to enable us to apply our equations to the values found for the lowering of pitch. Koláček[1] states that for a horseshoe magnet 340 mm. long, 12.5 mm. wide and 46 mm. thick vibrating in water the value of $\frac{T'}{T}$ was 1.03. Calculated from equations (d) and (e) the interval is 1.02.

[1] Loc. cit.

It was thought worth while to test equation (d) by applying it to the numbers found by Miss Laird, taking $\frac{w}{t}$ as equal to 1. It was found that the values thus determined for the interval of lowering in water agree with the observed values even better than do the values calculated on the simple supposition that the wire is loaded with the mass of the displaced fluid. For denser liquids, however, equation (d) does not agree so well as the equation of Stokes, since the lowering was found to depend on the pitch. The following table contains the frequencies in air and in liquid observed by Miss Laird, and the differences between the observed frequency in liquid and numbers calculated (1) from Stokes' equation, (2) from equation (d), and (3) on the assumption that the wire is loaded with the mass of the displaced liquid. These differences are called Δ_1, Δ_2 and Δ_3 respectively.

TABLE XII.

	Pitch in Air.	Pitch in Liquid.	Δ_1	Δ_2	Δ_3
Water.	73.8	70.1	− 2.9	− .4	− .6
	92.3	87.9	− 3.6	− .7	− 1.0
	105.5	99.4	− 2.8	+ .3	− .1
	113.4	107.7	− 3.8	− .5	− .9
	126.5	117.8	− 1.8	+ 1.7	+ 1.3
Solution of Pot. Carbonate.	73.8	65.1	− .8	+ 2.9	
	92.3	82.6	− 1.7	+ 2.4	
	105.5	93.9	− 1.2	+ 3.3	
	113.4	99.7	+ .1	+ 4.8	
	126.5	111.6	− .2	+ 5.2	
Mercury.	73.8	43.0	− 1.8	+ 2.8	
	92.3	53.3	− 1.3	+ 4.0	
	105.5	61.9	− 2.2	+ 3.5	
	113.4	65.8	− 1.5	+ 4.7	
	126.5	73.4	− 1.5	+ 5.2	

An attempt was made to see what would come from assuming that there was always attached symmetrically to the rod when in motion a constant mass of liquid of elliptical cross section. It will be seen from the way in which the quantity R was introduced that it is the ratio of the volume of this attached liquid to the volume of

the rod. Using Table IX. it was found that if the major axis of the ellipse were $2w$ and the minor axis $t + \frac{1}{2}w$, the periods calculated, on the assumption (used in this paper) that the lowering of pitch is due only to the added mass of liquid, agreed very well with the observed values.

PHYSICAL LABORATORY, BRYN MAWR COLLEGE,
May, 1901.

POYNTING'S THEOREM AND THE DISTRIBUTION OF ELECTRIC FIELD INSIDE AND OUTSIDE OF A CONDUCTOR CARRYING ELECTRIC CURRENT.

By William S. Franklin.

Introduction.

PROFESSOR J. H. POYNTING has derived,[1] from Maxwell's equations, an expression for the flow of energy in the electromagnetic field. According to Poynting the energy stream is, at each point, perpendicular both to the electric field and to the magnetic field, and the amount of energy which in one second streams across an area of one square centimeter at right angles to the stream is equal to $\frac{1}{8\pi} EH$, where E is the electric field intensity and H is the magnetic field intensity. J. J. Thomson[2] has pointed out that, although Poynting's expression for the energy stream is perhaps the simplest expression that will satisfy the necessary conditions, it is not the only expression that will do so, inasmuch as any energy flow which has no convergence, or in which the energy streams are reëntrant, may be superposed upon the Poynting stream without changing the distribution of energy which actually takes place in any given case.

In his paper above referred to Poynting gives a number of examples illustrating his theorem. Thus he describes, in a general way, the flow of energy from the dielectric of a charged condenser to the various parts of a wire through which the condenser is discharged; he describes the flow of energy from the seat[3] of the

[1] Phil. Trans., Vol. 175, Part II., p. 343, 1884.

[2] Recent Researches, p. 313.

[3] The author, for the purpose of his discussion, assumes this seat to be mainly at the surface of contact of acid and zinc.

electromotive force in a voltaic cell to the various parts of the circuit to which the cell delivers current; and he describes the flow of energy from the hot junction of a thermoelement, showing how an excess of energy flows in at the cold junction thus producing the heating effect discovered by Peltier.

Poynting's derivation of his theorem is entirely satisfactory and convincing to one who is familiar with the use of differential equations in the discussion of electromagnetic theory, and the only uncertainty which would appear to affect the result is the uncertainty of the fundamental truth of Maxwell's equations.

The examples given by Poynting are not, however, entirely satisfactory, and for a sufficient reason. An example of the theorem is, in the nature of things, a solution of Maxwell's equations for a particular case. Such a solution requires a complete specification of all conditions at the boundary of the region in which the solution applies and a complete knowledge of all singularities occurring within this region. One who appreciates this fact is likely to be satisfied with the examples given by Poynting, notwithstanding their lack of numerical detail, for the solution of Maxwell's equations for any ordinary current generator and its circuit is impracticable.

In a series of short articles appearing at intervals from 1897 to date, Mr. P. S. Wedell-Wedellsborg[1] subjects Maxwell's equations in general and Poynting's theorem in particular to criticisms which have suggested to the writer the desirability of supplementing Poynting's original discussion in three particulars, namely, by stating as clearly as possible the conditions which determine the electric field distribution inside and outside of a conductor carrying an electric current, by working out the numerical details of several simple illustrative examples, and by calling attention to a mechanical model, due originally to Maxwell, which not only gives a good representation of an energy stream in an electro-magnetic field, but gives vivid

[1] "Notiz ueber Poynting's Theorem." Zeit. f. Phys. Chem., Vol. 22, p. 222.

"Ueber die Gültigkeit der Maxwellschen Gleichungen." Zeit. f. Phys. Chem., Vol. 24, p. 367.

"Antwort an Herrn Scheye." Zeit. f. Phys. Chem., Vol. 29, p. 494.

"Widerlegung eines sehr allgemeinen und wichtigen Safzes der modernen Elektrizitätslehre." Zeit. f. Phys. Chem., Vol. 33, p. 631.

"Ueber Poynting's Theorem." Zeit. f. Phys. Chem., Vol. 35, p. 604.

pictures of the steady electric current and of the action of an electric oscillator with the accompanying energy fluxes. The writer is of the opinion, however, that an understanding of Poynting's theorem and a conviction of its truth can be based only upon Maxwell's differential equations, and that particular solutions are not only generally impracticable, but useless as well. Still the discussion given in the following pages may be of interest to those who are pleased only when a mathematical discussion concludes with particular integrals and concrete results.

The earlier criticisms of Wedell-Wedellsborg are met, in a manner that seems to the writer satisfactory, by D. A. Goldhammer,[1] and the need of a numerical solution of Maxwell's equations for a particular case involving the flow of energy into a conductor is well met by G. Mie[2] who gives a solution for the case of two long parallel cylindrical conductors carrying equal and opposite currents.

Wedell-Wedellsborg's criticisms are quite largely based upon the question of the distribution of electric field inside and outside of the current conductor. In regard to this matter, A. Scheye[3] points out that in most ordinary cases the effect of the energy of the electric field is very small and is ignored in the theory of current induction. In a later article Scheye[4] meets two explicit objections raised by Wedell-Wedellsborg, namely, that the electric field distributions required by Maxwell's equations do not agree with experiment, and that, granting the existence of this electric field we are led to inconsistent results. Scheye agrees with the first objection in so far as to admit that further decisive experiments are required. The second objection he combats, pointing out that inasmuch as Maxwell's equations are consistent among themselves they cannot lead to inconsistent results if no foreign assumptions are introduced.

[1] "Einige Bemerkungen über die von Hernn Wedell-Wedellsborg vermütete Nichtgültigkeit der Maxwellschen Gleichungen für das Innere der Konduktoren." Zeit. f. Phys. Chem., Vol. 23, p. 686.

[2] "Ein Beispiel zum Poyntingschen Theorem." Zeit. f. Phys. Chem., Vol. 34, p. 522.

[3] Zeit. f. Phys. Chem., Vol. 26, p. 159.

[4] Zeit. f. Phys. Chem., Vol. 32, p. 145.

ON ELECTRIC FIELD DISTRIBUTION.

Electric Field in the Conductor.—The electric field intensity at a point in a homogeneous conductor, of which the specific resistance is r, is equal to ri when i is the electric current density at the point. From this expression we can draw two important inferences, namely: (*a*) There is no free electric charge anywhere in the interior of a conductor carrying a steady electric current, for a steady electric current has no convergence,[1] and therefore the field intensity ri has no convergence if r is constant. (*b*) Near the boundary of the conductor the electric field in the conductor is parallel to the surface of the conductor, for i is parallel to the surface of the conductor, and so also is ri.

Electric Field Outside of the Conductor.—The distribution of electric field outside of the conductor is determined by certain conditions which must be satisfied at the surface of the current conductor and at the surfaces of other conductors in the region. At each point of the surface of the current conductor the tangential component of the external electric field must be equal and parallel to ri inside the surface, and at each point of the surfaces of other conductors the tangential component of the external field must be zero.

An immediate result of these conditions is that the electric lines of force are never normal to the surface of a current conductor, and that the electric lines of force may be parallel to the surface of the current conductor. In fact, however, it is only in exceptional cases that the electric lines of force are parallel to current conductors. Thus, in the particular case worked out by G. Mie[2] the electric lines of force are very nearly perpendicular to the surfaces of the current conductors, and in one of the examples given below the lines of force are parallel to the surface of the current conductor. There is always at least one point on any electrical circuit where the electric lines of force are parallel to the current conductor, this is the point where the conductor has no surface charge.

[1] That is, $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$, when u, v, and w are the components of i.

[2] Reference given above.

Reproduction by Stationary Charges of the Electric Field Distribution around a Current Conductor.—The distribution of electric field in the region surrounding an electric current, including the distribution inside of the conductor, can be reproduced by an arrangement of stationary electric charges, the electric current being reduced to zero. Imagine a given current conductor to have its specific resistance increased indefinitely, the size and shape of the conductor, and the generator electromotive force being unchanged. Then ri remains unaltered while i approaches zero. When the limit, $r =$ infinity, is reached the current is zero, the distribution of electric field and of electric charge is the same as at first, the electric charges are evidently stationary, and the magnetic field and energy stream have vanished. This ideal change of specific resistance of the current conductor might be realized by cutting the conductor into a very great number of lamina perpendicular to the lines of flow of the current, all the cross-cuts being made simultaneously.

That the electric charge which is distributed over a current conductor is stationary is further shown by the fact that these charges are surface distributions so that their movement would constitute current sheets in the surface of the conductor and there is no experimental evidence of the existence of such surface currents.

The Conception of Moving Lines of Force.—The idea of the side motion of the lines of force of the electric field surrounding a current conductor has been developed by Poynting,[1] by J. J. Thomson,[2] and by Wien,[3] and this idea has been extensively used by other writers to give concreteness to discussions of electromagnetic action. The writer is of opinion that the fundamental notions of the partial derivatives with respect to time and space, such as form the basis of Maxwell's equations, are quite adequate to the full description of the phenomena of electromagnetic action; and that the only advantage of the idea of the side motion of the lines of

[1] "The Connection between Electric Current and the Electric and Magnetic Induction in the Surrounding Field." Phil. Trans., Vol. 176, p. 277.

[2] Recent Researches, pp. 1–50.

[3] "Ueber den Begriff der Localizirung der Energie." Wied. Ann., Vol. 45, p. 685.

"Ueber die Bewegungen der Kraftlinien im electromagnetischen Felde." Wied. Ann., Vol. 46, p. 352.

force is its concreteness. In the case of a plane electromagnetic wave, for example, it seems to the writer to be quite as satisfactory to consider the rates of growth or decay of the electric and magnetic field intensities at a point in their relation to the space variations of these intensities in the neighborhood of the point, as to imagine an actual side motion of the lines of force although the latter idea is certainly better adapted to descriptive purposes. Either method appears to be applicable to any case inasmuch as Webster[1] has recently derived Maxwell's equations from a consideration of the side motion of lines of force.

In applying the idea of the side motion of the lines of force to the case of a steady electric current a difficulty arises. The charges in which the electric lines of force terminate are stationary and it seems that the lines of force also must be stationary. When the electric lines of force are parallel to the surface of the current conductor their side motion involves no immediate inconsistency, but these lines of force may in every case be traced to termini in stationary charges at some more or less remote part of the current conductor.

In the case of unsteady electric currents, as for example, the transmission of electric waves along wires,[2] the idea of side motion of the lines of force does not seem to lead to inconsistencies, for in this case the charges are not stationary.

Neglect of Electrical Energy in Ordinary Current Problems.—The energy associated with an electric current is in part the energy of the magnetic field and in part the energy of the electric field. In many cases the electrical energy is negligible in comparison with the magnetic energy. Thus, at a point distant one centimeter from a long copper wire one millimeter in diameter carrying a current of ten ampères, the magnetic energy is about 0.16 erg per cc. and the electrical energy is about 2×10^{-12} ergs per cc.[3] This example is somewhat misleading inasmuch as the magnetic energy falls off

[1] In a paper read before the American Physical Society, December, 1899.

[2] See paper by A. Sommerfeld, Wied. Ann., Vol. 67, p. 233. A number of diagrams are given in this paper from which a student can get a good idea of the trend of the electric and magnetic lines of force near a wire over which an electromagnetic wave is passing.

[3] This expression for the electrical energy is calculated from the component of the electric field parallel to the wire.

rapidly as the distance from the wire increases while, in case of a long straight wire, the electrical energy does not fall off. This is equivalent to the statement that for short circuits the electrical energy is negligible while for extended circuits, such as long distance transmission lines, the total electrical energy may be as great as or even greater than the total magnetic energy.

Experimental Evidence of Electrical Field Surrounding an Electric Current.—The writer is not inclined to agree with Scheye[1] that the experimental evidence as to the character of the electric field surrounding an electric current is to any considerable extent deficient. Ohm's law itself shows that the potential gradient parallel to a wire is equal to ri and this law is verified by a great variety of experimental evidence. On the other hand, the so-called capacity effects on long transmission lines and in submarine cables are due immediately to the component of the electric field at right angles to the surface of the conductors; all practical calculations of these capacity effects are based directly or indirectly upon Maxwell's equations, and these calculations agree with experiment as closely as could be expected when we consider first that such calculations are usually abbreviated and that dielectrics and magnetic substances deviate from that beautiful simplicity of behavior which we usually assume as a basis of our calculations.

The submarine cable tests by Crehore and Squier[2] constitute, perhaps, the most complete experimental verification, hitherto published, of the theory of the electric circuit involving both electric and magnetic energy.

Examples Illustrating Poynting's Theorem.

The following solutions of Maxwell's equations, illustrative of Poynting's Theorem, apply to two dimensions, so that the respective diagrams represent the solutions completely. The shapes of the conductors have been chosen with a view to simplicity and the electric lines of force and the energy stream lines belong to families of curves with which everyone is thoroughly familiar. A minimum of

[1] Reference given above.

[2] Presented in a paper read before the annual meeting of the American Institute of Electrical Engineers in May, 1900.

effort of the imagination is therefore required for a full comprehension of these solutions and their value, as examples of Poynting's theorem is correspondingly great.

The Slow Discharge of Two Parallel Charged Cylindrical Conductors through a Cylindrical Current Sheet.—The only case, exemplifying Poynting's theorem, for which a complete solution of Maxwell's equations can be set up for all space, is the case of the discharge of a condenser. This problem has been treated in a variety of cases by Hertz[1] and by J. J. Thomson.[2] These authors have treated mainly the case of the oscillatory discharge. The case of very slow discharge is of course much simpler. This case is described in a general way by Poynting[3] as an example illustrating his theorem. The following is an exact solution of the problem of the slow discharge of two parallel charged cylinders through a cylindrical current sheet.

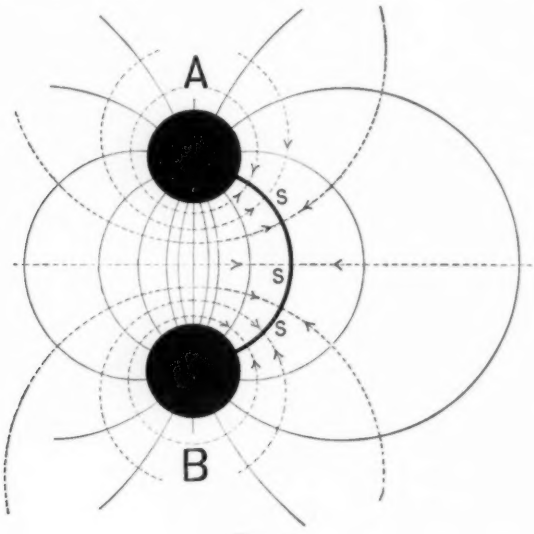

Fig. 1.

A and *B*, Fig. 1, represent two cylinders which are oppositely charged, and connected by a very high resistance sheet of metal *SSS*,

[1] "Die Kräfte der electrischen Schwingungen, behandelt nach der Maxwell'schen Theorie." Wied. Ann., Vol. 36, p. 1. A paper dealing with an oscillating electrical doublet, the simplest case, perhaps, of a discharging condenser.

[2] Recent Researches, pp. 251–387. The author treats of several types of electrical oscillators of finite dimensions.

[3] Phil. Trans., Vol. 175, Part II., p 351.

through which the cylinders are slowly discharged. The charges, which are uniformly distributed on the cylinders A and B, produce an electric field of which the lines of force are circles, as represented by the full lines in the figure. If this electric field due to the charges on A and B happens to satisfy the necessary boundary conditions along the current sheet SSS, then this field alone will constitute the solution of the problem under consideration. If this field does not satisfy the boundary conditions along SSS, then charge will be distributed over the two faces of the sheet SSS in such a manner as to modify the electric field distribution and satisfy the boundary conditions along SSS. Since we are seeking the simplest possible solution we will assume that the resistivity r, of the sheet SSS per unit width (perpendicular to the plane of the paper) varies from point to point in such a way that ri at each point of SSS is equal and parallel to the electric field at that point due to the charges on A and B, where i is the current per unit width of the sheet SSS. In this case the electric field due to the charges on A and B satisfies all necessary conditions, and the electric lines of force are circles touching A and B orthogonally. The magnetic field[1] is everywhere perpendicular to the plane of the paper and the energy stream lines are circles cutting the electric lines of force orthogonally.

Current in a Cylindrical Sheet, Nearly Closed.—The above ideal example may be realized for the case of a finite electric current flow-

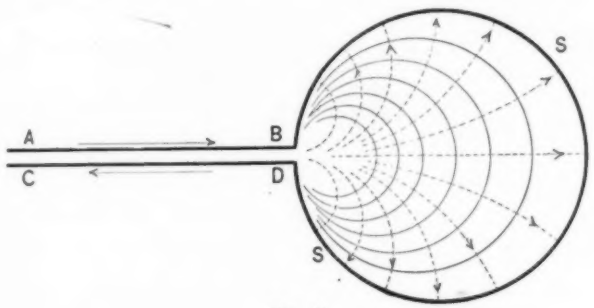

Fig. 2.

ing steadily in a cylindrical sheet as follows: Two plane current sheets AB and CD, Fig. 2, are very near together and they separate

[1] The magnetic field is not everywhere of the same intensity. In fact it grows less and less intense at increasing distances from the current sheet. The energy stream has divergence.

at *BD* forming a cylindrical shell *SS*. The electric current flows as indicated by the arrows and it is required to find the distribution of the electric field and of the energy stream inside of the shell. In this case the region in which the field distribution is to be determined is not a closed region, that is, it is not completely shut off by a conducting screen from the region occupied by the electric generator but since *AB* and *CD* are very near together the boundary conditions across the gap *BD* may be represented by an electric doublet (line doublet). The problem is therefore determinate whatever the shape of *SS* and whatever the manner of variation of the resistivity of the shell from point to point. The simplest case is that in which *SS* is a circular cylinder. The lines of force from a linear doublet are circles and if *ri* at each point of the shell is equal to the field intensity at that point due to the doublet at *BD* then the boundary conditions are completely satisfied by the electric field due to the doublet alone and this, therefore, will be the actual electric field inside of *SS*. In this case there is of course no electric charge on the inner surface of *SS*. The electric lines of force are represented by the full line circles in Fig. 2 and the energy stream lines are represented by the dotted circles. These lines are not extended into the region immediately adjacent to the gap *BD* inasmuch as the solution under consideration does not hold near *BD*. The magnetic field inside of *SS* is uniform and the energy stream has no convergence.

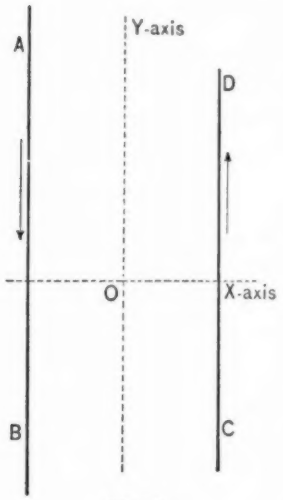

Fig. 3.

Electric Current in Two Parallel Infinite Conducting Sheets.—Electric current flows down an infinite plane conducting sheet *AB* and up an infinite conducting sheet *CD* parallel to *AB*. The current per unit width in *AB* and *CD* is *i*, and the resistivity of each sheet per unit width is *r*. Required the distribution of the electric field and of the energy stream in the region between *AB* and *CD*.

Choose the *Y*-axis midway between the planes and parallel to the electric currents. Choose the *X*-axis so

that it cuts the two planes at points which are at the same potential. Let X and Y be the components of the electric field at any point. Then

$$\frac{dX}{dx} + \frac{dY}{dy} = 0 \tag{1}$$

expresses the fact that the convergence of the electric field is zero throughout the region between AB and CD.

The magnetic field is everywhere constant so that

$$\frac{dY}{dx} - \frac{dX}{dy} = 0. \tag{2}$$

From equations (1) and (2) we have:

$$\frac{d^2X}{dx^2} + \frac{d^2X}{dy^2} = 0 \tag{3}$$

and

$$\frac{d^2Y}{dx^2} + \frac{d^2Y}{dy^2} = 0. \tag{4}$$

These equations permit separate solutions to be set up for X and for Y.

In order that the boundary conditions for X and Y may be expressed as simply as possible let the distance between the planes AB and CD be taken as two units of length, so that the abscissas of the planes will be -1 and $+1$ respectively, and let the value of ri be represented by the letter a.

The boundary conditions for Y are:

$$\begin{aligned} Y &= -a \text{ when } x = -1, \\ Y &= +a \text{ when } x = +1. \end{aligned} \tag{5}$$

The solution of equation (4) under these conditions corresponds to the steady flow of heat (Y being temperature) through a homogeneous substance between two infinite planes AB and CD, these planes being at temperatures $-a$ and $+a$ respectively. Therefore, Y, is a linear function of x which has the prescribed values $\pm a$ when $x = \pm 1$. Therefore:

$$Y = ax. \tag{6}$$

The boundary conditions for X are determined with the help of

equation (1) as follows: Y being constant along AB and along CD, $\frac{dY}{dy}$ is zero along these planes. Therefore from equation (1) $\frac{dX}{dx}$ is zero along these planes. That is:

$$\frac{dX}{dx} = 0 \text{ when } x = \pm 1 \tag{7}$$

The solution of equation (3) under these conditions corresponds to the steady flow of heat (X being temperature) through a homogeneous substance between two thermally insulating planes AB and CD. Therefore X is a linear function of y. That is:

$$X = by + c \tag{8}$$

Equations (1) and (2) are not necessarily satisfied by solutions of (3) and (4). Equation (1) however has been used in finding the boundary conditions of X from the given boundary conditions of Y, so that equation (1) is everywhere satisfied.[1]

The condition imposed by equation (2) fixes the value of b, equation (8). Thus, from (6) we have $\frac{dY}{dx} = a$ whence by equation (2) we have $\frac{dX}{dy} = a$; but by equation (8) $\frac{dX}{dy} = b$. Therefore $b = a$. The value of the constant c, equation (8), is determined as follows: The difference of potential between AB and CD at a distance y above the X-axis is equal to $2X$, or by equation (8) it is equal to $2by + 2c$, since the distance between AB and CD is 2 units. Therefore for $y = 0$, this potential difference is $2c$, but the X-axis has been chosen so as to pass through equipotential points on AB and CD, so that $c = 0$, and equation (8) becomes:

$$X = ay. \tag{9}$$

The differential equation of a line of force of the electric field is:

[1] The two continuous functions X and Y satisfy equation (1) over the entire boundary to the region between AB and CD. Therefore, according to the theory of functions, they must satisfy this equation throughout the region. The small number of disposable constants in the above solutions of equations (3) and (4) is due to the fact that the boundary conditions are fully considered in the thinking out of the solutions by the aid of the heat-flow analogy.

$$\frac{dy}{dx} = \frac{Y}{X}$$

or, using equations (6) and (9),

$$\frac{dy}{dx} = \frac{x}{y}$$

of which the integral is

$$x^2 - y^2 = C. \qquad (10)$$

The lines of force are, therefore, equilateral hyperbolas of which the asymptotes are the two straight lines ($y = x$) and ($y = -x$).

The magnetic field in the region between AB and CD is uniform and perpendicular to the plane of the paper in Fig. 3. The energy stream lines are, therefore, in planes parallel to the paper in Fig. 3 and since these stream lines are perpendicular to the electric field, the differential equation to a stream line is:

$$\frac{dy}{dx} = -\frac{X}{Y}$$

or, using equations (6) and (9),

$$\frac{dy}{dx} = -\frac{y}{x}$$

of which the integral is

$$xy = C. \qquad (11)$$

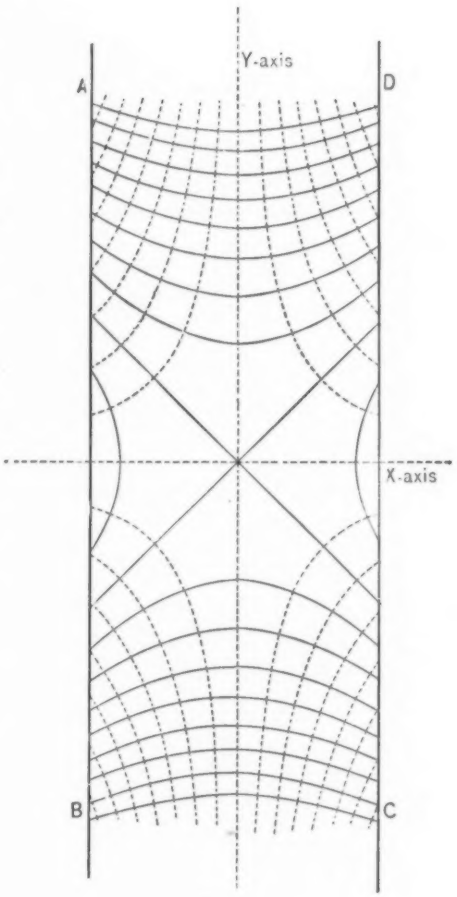

Fig. 4.

The energy stream lines are, therefore, equilateral hyperbolas of which the axes of reference are the asymptotes.

Fig. 4 shows the electric lines of force and the energy stream lines. It is evident that the electric lines of force touch the conducting surfaces obliquely, except in the immediate neighborhood of the two points where the planes are cut by the X-axis. At these points the electric field near the planes is parallel to the planes, and the planes have no surface charge.

The above solution, as shown in Fig. 4, holds for a variety of conditions as follows:

1. Either conducting plane AB or CD may be replaced by another parallel plane at any distance x' from the Y-axis, provided the resistivity of the new plane is: $r = \frac{ax}{i}$; for, the value of Y along the new plane, as given by equation (6), will be the value required by the boundary conditions, namely $Y = ri$.

2. A conducting plane, parallel to the X-axis and perpendicular to the paper in Fig. 3, may be connected across from AB to CD at any distance y' from the X-axis and the flow of current changed so as to come down on AB to the new plane, thence across to CD, and up on CD, provided the resistivity of the new plane is $r = \frac{ay'}{i}$; for the value of X along the new plane, as given by equation (9), will be the value required by the boundary conditions, namely $X = ri$.

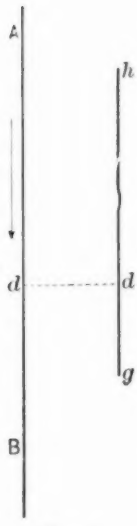

Fig. 5.

3. A conducting sheet of metal may be placed in the position ofany one of the equipotential surfaces (indicated by the energy stream lines in Fig. 4) without in any way disturbing the electric field, the magnetic field, or the energy stream. In particular a plane sheet of metal may be placed midway between AB and CD without producing any disturbance. This fact is made use of in a discussion, which follows, of the effect of a metal shield near a current conductor.

Flow of Energy Around a Metal Shield Placed near a Current Sheet.—An insulated metal plate gh, Fig. 5, is placed as a shield near and parallel to a plane current sheet AB. The potential of the shield is equal to the potential at the point d on AB, that is, the

shield may be supposed to be connected to AB by a wire dd. Required the distribution of the electric field and of the energy stream between the shield and the current sheet.

The required distribution is represented by the portion of Fig. 4, which is between AB and the Y-axis, the line dd, Fig. 5, being on the X-axis.

If the potential of the shield is equal to the potential of a remote point on AB, then the distribution of electric field and of energy stream between AB and gh will be represented by a portion of Fig. 4 between AB and the Y-axis and entirely above or entirely below the X-axis. The distribution of electric field near the edges of gh will be similar to the distribution of electric field near the edges of a small plate which is near a large plate, forming an ordinary air condenser. The character of the electric field in this case is, in a general way familiar to the reader, and the energy stream lines are of course perpendicular to the electric lines of force, so that the energy stream lines diverge into the surrounding open region near the edges of the plate gh, Fig. 5.

When the potential of gh is equal to the potential of AB at a point between g and h the energy stream enters the shielded region at g and at h, otherwise the energy streams into the shielded region at one end only, a portion of this energy flows into the conducting sheet AB, and the remainder streams out at the other end.

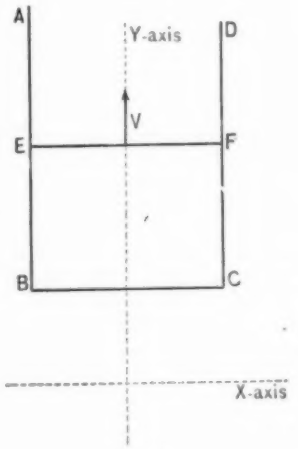

Fig. 6.

Flow of Energy in Case of an Induced Current.—AB, BC and CD, Fig. 6, are conducting planes, EF is a sliding plane in contact with AB and CD, and the entire region is a uniform magnetic field perpendicular to the plane of the paper. The sliding plane EF moves at low velocity V which increases[1] in proportion to its dis-

[1] This velocity is assumed to be small so that the field distribution may everywhere accommodate itself to the changing conditions without unsteadiness. The velocity would have to be at each instant a little in excess of the value required simply to overcome the increasing resistance of the circuit, on account of the fact that the inductance of the circuit is increasing also.

tance from BC so that the electromotive force induced in EF may suffice to maintain a constant current in the increasing circuit $EBCF$. It is required to find the distribution of the electric field and of the energy stream inside and outside of $ABCD$.

The curved lines in Fig. 7 represent the electric lines of force and the energy stream lines. The regions $AEFD$, $EBCF$, and the external region are treated separately.

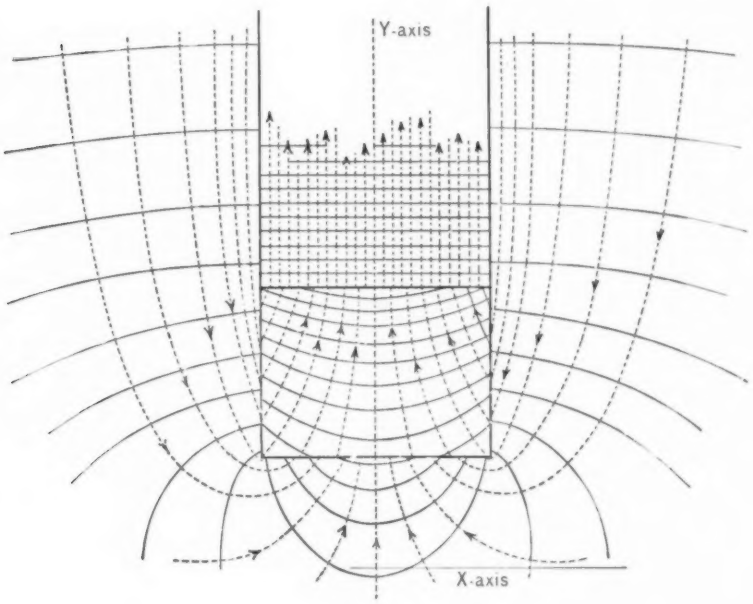

Fig. 7.

Let e be the electromotive force induced in EF, let r be the resistivity of the conducting planes per unit width, and let i be the current per unit width of planes. Let m be the length of EF, and let l be the length EB at the instant for which the solution is to be found. Let the axes of reference be chosen as shown in Fig. 6.

(*a*) In the region $AEFD$ the electric field is uniform and parallel to EF. The intensity of this field is $\left(\frac{e}{m} - ri\right)$; less than $\frac{e}{m}$ because of the loss of electromotive force in overcoming the resistance

of the slide EF. The energy stream in the region $AEFD$ is uniform and parallel to EA.

(b) The distribution of the electric field and of the energy stream in the region outside of $ABCD$ is approximately as shown by the lines in Fig. 7. The energy stream passing up between A and D returns, as indicated by the dotted lines, outside of AB and DC, and converges upon EB, BC and CF. A portion of this energy stream changes into heat in the conductor and the remainder flows through the conductor into the region $EBCF$.

(c) The distribution of the electric field and of the energy stream in the region $EBCF$ is exactly the same as that shown by a portion of Fig. 4 as indicated by the hyperbolic lines of force and lines of flow in Fig. 7, except that the energy stream is reversed in direction. The energy which streams up to EF in the region $EBCF$ exceeds the energy which streams away from EF in the region $AEFD$ by the amount $(mri^2 + 2lri^2)$ which is the equivalent of the heat appearing in EB, BC and CE, and is equal to $(P - mri^2)$, where P is the total power expended in moving the slides EF and mri^2 is the power equivalent of the heat appearing in EF.

The total fluxes of energy in the various energy streams above described may be calculated independently of a precise algebraic solution of the electric field distributions and the entire consistency of the above statements is easily shown.

The statements given above concerning the distribution of electric field and of energy stream in the two regions $AEFD$ and $EBCF$ are rigorously true as is easily seen from foregoing discussions given in this paper. On the other hand, the distribution in the region outside of $ABCD$ is only approximately represented in Fig. 7. The building up of algebraic functions for the electric field components X and Y in the external region so as to satisfy the boundary conditions over $ABCD$ is easily reduced to well known and soluble cases of heat flow, but the developments are more complicated than would be warranted in the present paper which is intended to be as simple as possible.

South Bethlehem, Pa.,
April, 1901.

POLARIZATION AND INTERNAL RESISTANCE OF ELECTROLYTIC CELLS. II.

BY MARTIN D. ATKINS.

PART II. THE AREA OF ELECTRODES IN RELATION TO THE CONSTANT *K*.

HAVING determined, as it seemed, with a fair degree of certainty, that the formula under consideration, while not in all cases applicable to electrolytic action, was yet an approximate working formula for fresh zinc or copper electrodes in the solutions tested, attention was turned to the question of the relation of the constant K to

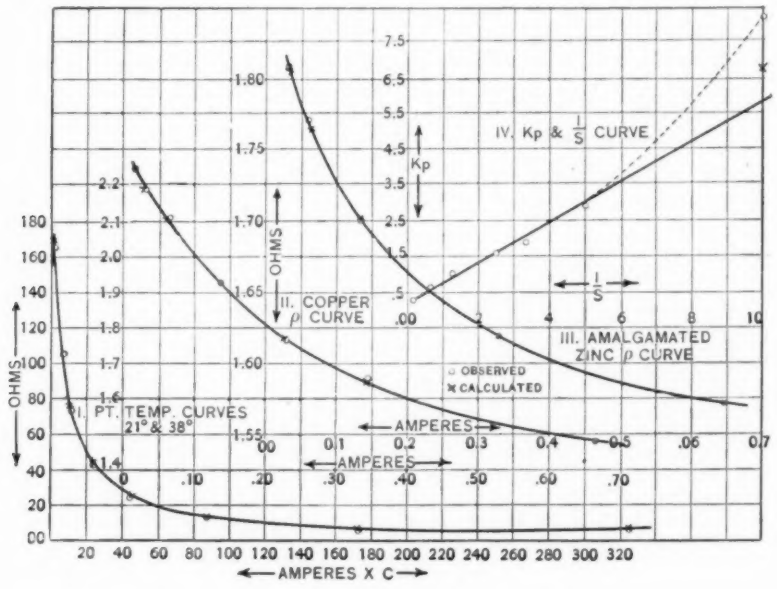

Plate III.

the area of the electrodes. The theory demands, as is seen from the substitution $K = \left(\frac{Bt}{S}\right)$, that K shall be inversely proportional to

the area of the electrode. It was felt that, could this point be determined, and should the results affirm such a relation, the conclusions already reached could be very much more confidently asserted. It may be said here, that the results, up to the present obtained, do not establish this relationship.

As has already been said, a large number of observations were made with zinc plates in zinc sulphate. The results proved conclusively that to obtain like conditions by amalgamation was very difficult, if not impossible. Even when handled, as far as could be seen, in a precisely similar manner, widely variant results would be obtained with the same plates; giving no encouragement to hope for concordant results with plates with widely varying areas. While very suggestive results were often obtained, it was found impossible to repeat them with certainty; and hence they can remain only as

TABLE XIX.

Areas.	*K*	*P*	*KP*
1	5	.025	.13
2	11	.058	.64
4	20	.052	1.04
8	41.3	.053	2.29

suggestive. Such a series of results is given below.[1] It should be understood that these are not results selected from a large number; but a series, taken under conditions as nearly like as could be arranged: The objection to basing any argument upon them, lies in the fact that they could not be repeated, even when, as far as could be seen, like conditions were realized.

TABLE XX.

Areas.	*K*	*P*	*KP*
1	9.9	.034	.34
2	9.4	.065	.61
4	16.7	.060	1.00

Another series, taken with plates amalgamated and allowed to stand in air four hours and then used, gave a set of curves which

[1] Tables XIX.–XX.

cut off from the y axis values, corresponding to $KP = \rho$ for $I = 0$, as follows:

TABLE XXI.

Zinc, Amalgamated, Dry. Jan. 11. T, 19°-20°.

Dimensions.	Areas (relative).	Range of App. Res., ρ	Differences, *i. e.*, $KP = (\rho - r)_{i=0}$
10 x 13 cm.	8	1.23 - 1.16	.07
10 x 6.5 "	4	1.99 - 1.83	.16
5 x 6.5 "	2	3.00 - 2.68	.32
5 x 3.25 "	1	5.34 - 4.68	.66

While these coincidences, if accidental, are certainly remarkable, the curves are so irregular, and the total changes so small, that the computation of K and P, from values so near each other, and giving differences in themselves so extremely small, gave no satisfactory results.

Most nearly satisfactory conditions having been obtained, for securing like results, by flowing the surface of the plates with pure mercury, the thought suggested itself that, perhaps, liquid electrodes would insure the condition of like surfaces, necessary for our purpose. Small cups, with areas as 1 : 3 : 5, were, therefore, prepared from short sections of glass tubing, and these were filled with an amalgam containing a small amount of C. P. zinc. Using these as electrodes, in the solution before employed, gave us again very suggestive results; which, however, owing perhaps to solution changes, could not be repeated.[1] It is the intention to soon repeat these observations, the solution having been left standing four months over the amalgam, with the hope that the irregularities might adjust themselves.[2] The results follow.

TABLE XXII.

Areas (relative).	Observation 1		2		3	
	K	P	K	P	K	P
1	60.6	.290	55.6	.375	42.3	.224
3[1]					18.9	.250
5	12.6	.119	10.15	.220	14.6	.190

[1] Used only in last observation.

[2] The repetition of these measurements since made indicates changes still going on, and is equally inconclusive.

There was an advantage in using these electrodes over any others employed, viz., their rate of change with the time alone was very small, as may be seen by reference to their time curve, Plate I, No. III. Comparison with the zinc in the form of plates, even when best amalgamated, shows a much slower rate of increase of the apparent resistance on the part of the Hg — Zn electrodes. Even the copper, later used, was more rapid in its time change, as its time curve shows, Plate I., No. IV. This fact, together with the suggestive relations found for K, is sufficient to encourage to a further investigation of this combination.

Cu in $CuSO_4$.

These observations with the zinc proving far from satisfactory, we returned to the use of copper electrodes in the copper sulphate solution before employed.

Inspection of the Bartoli-Wiedeburg formula (1) in its original form,

$$P = \frac{P_1}{2}(1 - e^{-\frac{\beta_1 q}{s}}) + \frac{P_2}{2}(1 - e^{-\frac{\beta_2 q}{s_2}}),$$

shows that, if one electrode be made very large in comparison with the the other, the same form results, as from the equalities of areas, and the assumption that β_1 equals β_2,

$$i. e., p = P(1 - e^{-KI}), \text{ as before.}$$

To determine which electrode could be most advantageously eliminated in this manner, the following tests were made. First, a large kathode was used and the anode varied, and then, *vice versa*. The results indicated that the polarization was practically not affected by varying the sizes of the anode, so long as they were not too small. On the other hand, change in the area of the kathode was attended always with change in the values of K and P. Table XXIII. contains the results of these tests.

With the purpose of rendering the anodic polarization negligible in all cases, an anode of 12 plates was constructed. These plates furnished a surface 24 times that of the largest kathode, and over 1,200 times that of the smallest. A large battery jar, containing ten

TABLE XXIII.

Cu in $CuSO_4$. 100 *G. in* 1 *L.*

Varying Anode.				Varying Kathode			
Relative Areas.		Constants.		Relative Areas.		Constants.	
Kathode.	Anode.	*P*	*K*	Kathode.	Anode.	*P*	*K*
	1	.066	18	1		.155	16?
	2	.054	14	2		.105	21
24	8	.047	11	8	24	.049	18
	16	.043	10	16		.046	11
	24	.041	9	24		.039	10

liters of solution, was used; and the cell thus constructed, placed in a crock filled with ice water. The temperature thus secured was about 8° C.

The time curve of this cell has already been referred to. The low temperatures retarded the change somewhat, although not so much as it was hoped they might.

As with the zinc, the first experiments aimed at securing surfaces always of the same character. To attain this end was found, as had been the case there, a matter of extreme difficulty. The plates were taken to the cell from acids, from various grades of emery paper, from the polishing wheel, alike with indifferent results. They were placed in the cell from air, alcohol, water, and the solution to be tested, with results that left much, in the way of constant values, to be desired. As a final attempt the plates were electroplated with the same current density for the same times. But, even in this way, it was impossible to obtain like conditions of surface, for the greatly varying areas employed, and the large number of observations necessary.[1]

While the curves show values for KP for $I = 0$ very nearly in inverse proportion to the respective areas, the computed values are far from the desired agreement. While often a single observation in a series falls entirely out, there are, on the other hand, many cases where the values show a very near approach to those demanded by the theory. It is very evident that the disturbing influences are hard to control. Very slight changes in the value for P

[1] Cf. B. E. Moore, Pol. Copper Voltameter, PHYS. REV., Vol. X., 1900.

make very large differences in those for K. The value of P, itself, is so sensitive to surface conditions that it is doubtful if more conclusive results should be expected, until these conditions are better understood and controlled. Since varying currents must be used, it is evident that initial conditions of exact similarity could not remain such long. An attempt was made to return to the initial conditions by replating after each observation, but, owing to the time change doubtless, this resulted usually in an increase in the value of the constants obtained. The following tables contain the

TABLE XXIV.

$CuSO_4$, 100 G. in 1 L. Copper Anode of twelve Plates. Copper Kathodes of Areas from 1 to 64.

Relative Areas.	K	P	KP
2	66.5	.051	3.33
4	39.5	.043	1.69
8	15.9	.053	.84
16	22.2	.031	.67
64	6.3	.039	.25

TABLE XXV.

Relative Areas.	K	P	KP
1	91.5	.109	10.10
2	26.2	.090	2.36
3	40.2	.046	1.85
4	29.4	.049	1.44
8	15.8	.072	1.14
16	19.1	.025	.48
64	11.0	.016	.18

TABLE XXVI.

Combined Results from Table XXIV and XXV.

Areas.	K	P	KP	$\frac{I}{KP}$	$\frac{I}{KP}$ using .15.
1	91.5	.109	10.10	.10	.15
2	66.5	.051	3.33	.297	.30
3	40.2	.046	1.85	.55	.45
4	39.5	.043	1.69	.594	.60
8	15.9	.053	.84	1.18	1.20
16	19.1	.025	.48	2.12	2.40
64	11.0	.016	.18	5.77	9.60

results of two series of observations. No attempt has been made usually to completely solve these results for r. K and P have been computed by the approximate method before described. It was felt that too much uncertainty attended the conditions to warrant the large outlay of time and labor necessary to completely solve all these curves; while a sufficient number were solved, and

are given in Tables XXVII. to XXIX., inclusive, to prove that they are in agreement with the formulæ, as was to be expected from Part I. of this paper.

If we divide the observed values for KP in tables XXIV. and XXV. by the inverse areas of the plates we obtain from the first set the mean ratio .154; from the second .121; a mean of .138. Setting up a series upon this value, and taking the mean observed values, we have the results shown in the following table:

Obs. Mean. KP	Computed KP for mean .138.	$\frac{1 \times 10}{S}$
.22	.14	.16
.56	.55	.63
.99	1.10	1.25
1.57	2.20	2.50
1.85	2.84	3.30
2.85	4.40	5.00
8.38	8.80	10.00

If these values (observed) be now plotted as ordinates of a curve, which has as abscissæ the values $\frac{1}{S}$, they lie very nearly upon a right line, with the exception of the last. As this value is obtained from a single observation, and the value for P is in that very large,

TABLE XXVII.

Copper $\frac{2}{64}$. *Apr.* 18. $T = 8.0°$; $R = 10.0$ *ohms*. $K = 34.67$; $P = .062$; $KP = 2.15$.

Cap.	dC	dR	I	ρ	$(1 - e^{-KI})$	r	P
.05	135	136.5	.5348	9.89	1.0000	9.77	
.1	150	150.5	.2948	10.02	1.0000	9.81	.06315
.2	207.5	204	.1998	10.14	.9990	9.83	.06046
.3	197.5	189.5	.1237	10.34	.9863	9.84	.05904
.5	173.5	161	.06307	10.66	.8877	9.78	.06708
3.2	249	206	.01261	11.50	.3555	9.74	

it is quite possible that this point is too high. The curve is given in Plate III., No. IV.

Solved results. Copper electrodes in $CuSO_4$, from part II investigation.

Table XXVIII.

Copper $\frac{1}{32}$. *Apr.* 16. $T = 12.0°$; $R = 10.0$; $K = 52$; $P = .0405$; $KP = 2.08$.

Cap.	dC	dR	I	ρ	$(1 - e^{-KI})$	r	P
.05	153	132.5	.5191	11.55	1.0000	11.473	
.1	167.4	144	.2821	11.62	1.0000	.479	.04641
.2	238	204	.1998	11.67	1.0000	.471	.03038
.3	225	190	.1241	11.80	.9984	.480	.03979
.5	195	161.5	.0633	12.08	.9624	.474	.04165
1.0	209	164.5	.0322	12.52	.8128	.516	.03486
3.2	282	207.5	.0127	12.96	.4833	.445	.04565
3.2	182	129	.0079	13.18	.3368	.482	

Table XXIX.

Copp r $\frac{1}{4}$. *Mar.* 28. $T = 25.4°$; $R = 2.0$ *ohms.* $K = 7.10$; $P = .1480$; $KP = 1.05$.

Cap.	dC	dR	I	ρ	$(1 - e^{-KI})$	r	P
.5	252	340	.6650	1.474	.9911	1.249	
1.0	294	351.8	.3445	1.654	.9134	1.244	.1466
1.0	207	231.8	.2270	1.765	.8005	1.260	.1523
1.0	136	138	.1351	1.927	.6168	1.251	.1474
3.2	233.8	213	.0652	2.101	.3705	1.243	.1470
3.2	126	106	.0324	2.182	.2057	1.262	
3.2	68.3	52	.0159	2.242	.1068	1.253	

Supplementary Experiments.

Upon scrutinizing the whole field investigated, two results, worth especial consideration, were apparent. *First*, the marked increase in apparent resistance all the cells showed upon standing. *Second*, the current effect, or cutting down of the resistance on sending large currents through the cells.

Absorption of the ions of metal by the plates having been a suggested cause for the first of these phenomena, the following test was made. Plates prepared by silvering glass were used as electrodes; the thought being that the absorption within the material of the electrodes would, because of their microscopic thickness, be soon completed. So far as tested this was found to be the case. The apparent resistence increased rapidly upon first placing such electrodes in the solutions, becoming constant within the first quarter hour, and remaining thus for several hours—in some in-

stances 24. Unfortunately the films soon disintegrated, or scaled from the glass, and so conclusive data could not be gathered. These tests will be repeated.

An investigation was made as to the cause of the second, the current, change. With a highly polished, small copper electrode, the reading on the most sensitive point (smallest current), was taken. Then the plate was plated with a large current for ten minutes, readings being taken every 2 minutes. The curve for drop in ρ thus obtained is shown on Plate I., No. V. It will be noticed that the time change was entirely overcome in this experiment. It is probable that with the smallest plates used, and the largest currents, a single swing of the pendulum would cut the deflection down even while the current was passing. This influence would give us on the large currents and small plates, always relatively small values for P. This they are found to be, with those metals which are most susceptible to the current effect. By plating a short time with a small current, the deflections could be forced up to their original value. It is difficult to see how the error thus introduced can be eliminated.

PART III. TIME TESTS.

Although it was evident that the relationship of areas demanded by the Wiedeburg theory had failed of establishment, the further relationship which would demand that K be proportional to t was tested. By varying the positions of the keys, the time during which the current was allowed to flow through the cell was widely varied. In all cases such variation had no effect whatever upon the values of the deflections produced.[1]

PART IV. INITIAL POLARIZATION CAPACITY FROM OBSERVED DATA.

The applicability of this method to measuring the polarization capacity of an electrolytic cell has already been described.[2] Below

[1] Since this paper was written an article has been published by Mr. B. E. Moore, describing investigations upon the copper voltameter. In this report Mr. Moore has found that P increases as the time of current flow is increased. The time intervals used were however, very much shorter than any employed by this investigation. Vide, PHYSICAL REVIEW, January, 1900, pp. 47–49. "Polarization and Internal Resistance of the Copper Voltameter," by B. E. Moore.

[2] This article, p. 103, also 3.

are given several such values, for the cells employed in this research, computed from the formula [1]

$$C = \frac{t}{Kp}.$$

In all cases $t = 0.2$ second.

I. Zinc plates in $ZnSO_4$.

Area sq. cm.	Init. P. Cap. pr. 1 cm. sq.	
130	.00462	Mean .00529
65	.00482	
32.25	.00596	
16.125	.00577	

II. Copper plates in $CuSO_4$.

Area sq. cm.	Init. P. Cap. pr. 1 cm. sq.	
12.18	.0076	Mean .0062
12.18	.0078	
24.36	.0055	
48.72	.0040 (?)	

III. Silver plates in $AgNO_3$. One trial .000567 per 1 cm. sq.

IV. Platinum in H_2SO_4. One trial .0106 per 1 sq. cm.

A complete computation for all the data has not been attempted.

Conclusions.

A review of the results of this investigation justifies the following special conclusions:—

1. Electrodes of copper, in $CuSO_4$, and zinc (amalgamated, both plates and liquid amalgam), in $ZnSO_4$, when freshly placed in their respective solutions, give, with varying currents, curves completely satisfied by a formula of the Bartoli-Wiedeburg form, and its derived equations.

2. With such electrodes, therefore, the internal resistance r is, according to our assumption, constant.

3. Electrodes of silver in $AgNO_3$, and zinc in $ZnSO_4$, in the main, give curves satisfied by such a formula.

4. Electrodes of platinum (polished), in H_2SO_4, do not conform, as far as investigated, to the formulæ. The maximum value for

[1] Guthe and Atkins, Proceed. A.A.A.S., 1899, p. 109.

the polarization, however, is in agreement with that found by other methods.

5. Electrodes of all these metals in their respective solutions show upon standing a marked increase in apparent resistance. All results obtained concur in their testimony that this is not an increase in the real resistance of the cell.

6. Neither the inverse proportionality of the constant K to the area of the electrodes, nor its direct proportionality to the time of flow of current, is conclusively established by the results obtained.

While the theory of Wiedeburg is, therefore, not sustained as to its constants by the results of this research, an equation of the same general form would seem to be suggested as satisfying the curves for internal resistance and polarization of electrolytic cells.

GENERAL CONCLUSIONS.

(*a*) The method of this research furnishes a means of determining, to a degree of accuracy of 1 per cent., the internal resistance of electrolytic cells. Even those which do not admit of determination by the Kohlrausch method yield readily to this.

(*b*) The maximum value of the polarization, for a given combination of metal and solution, can be determined by this method.

(*c*) The method can be used for determining the polarization capacity of electrolytic cells even when this is very large.

In conclusion, I wish to express my sense of obligation to Professor Henry L. Carhart, who has placed at my disposal the facilities of his laboratories, for the purpose of this research. I wish also to avail myself of this opportunity to thank, most heartily, Dr. K. E. Guthe, with whom the investigation originated, for giving so generously of his personal direction and assistance to the details of the work at every stage.

UNIVERSITY OF MICHIGAN,
PHYSICAL LABORATORY, June, 1899.